Probability monads & stochastic dominance

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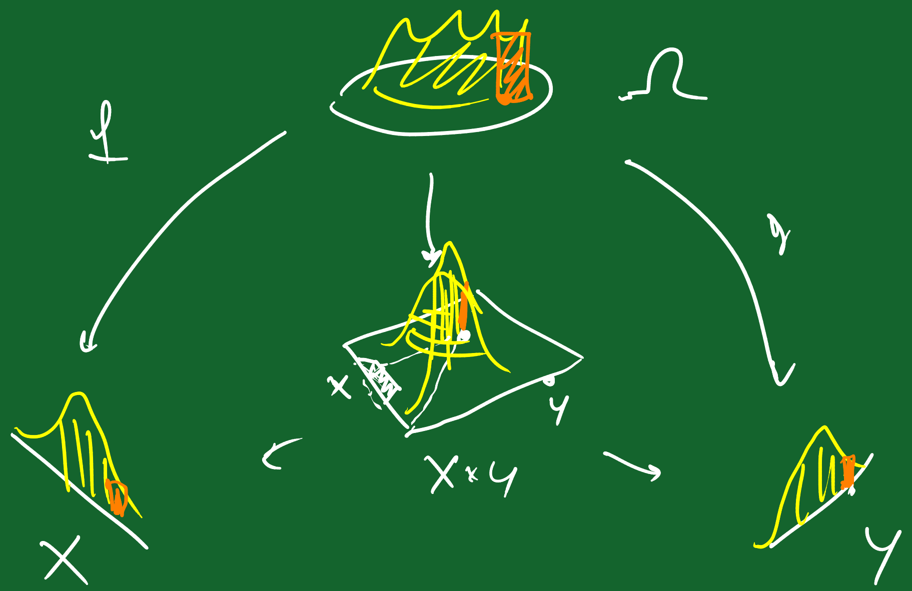
Stochastic dominance = compare prob. measures.

- How do we talk about this using monads?

PROB.

Random variables vs prob. measures.

$(\Omega, \mathcal{F}, \mu)$ $\qquad \mu \in P\Omega$



Meas, Giry monad.

$$\mu \longmapsto f_*\mu$$

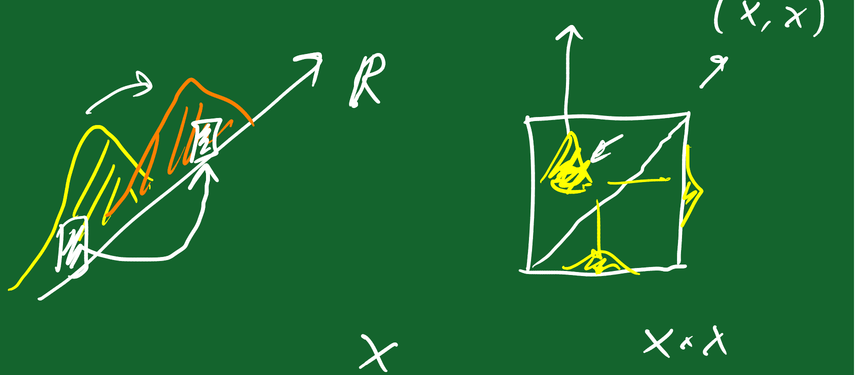
$$\begin{tikzcd} P\Omega \arrow[r, "f_*"] \arrow[d, "g_*"'] \arrow[dr, "(f,g)_*"] & PX \\ PY & P(X \times Y) \end{tikzcd}$$

$g_*\mu$

First-order stoch. dominance

"how much" but randomly. $\{\leq\} = \{(x,y) : x \leq y\}$
order relation

$(X, \leq)$ poset $\rightsquigarrow$ $(PX, \leq)$

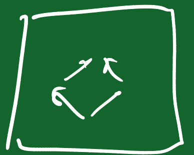


$$\Omega \overset{f}{\underset{g}{\rightrightarrows}} X \qquad \mu \in P\Omega$$

$f \leq g$ μ-almost surely (a.s.)

if $\underbrace{\{\omega \in \Omega \mid f(\omega) \leq g(\omega)\}}_{\text{has prob. 1.}} \supseteq$ meas. set of full meas.

$$\begin{tikzcd} & X \\ \Omega \arrow[ur] \arrow[r, "(f,g)_*"] \arrow[dr] & X \times X \\ & X \end{tikzcd}$$



We can move the mass from $f_*\mu$ to $g_*\mu$ along the arrows of the order.

Let now $p, q \in PX$.

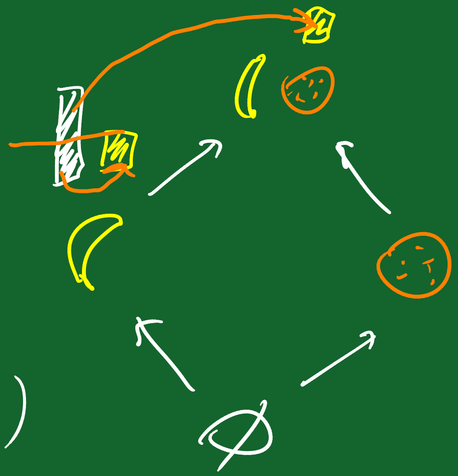
Def. We say $p \le q$ in the stochastic order

in 1st-order stochastic dominance

if (equivalently): distributions $\left(\Omega \xrightarrow{f} X \quad f_* \mu\right)$

- They are laws of RVs f, g with $f \le g$ a.s.
- They admit a joint which assigns full measure to the order relation.

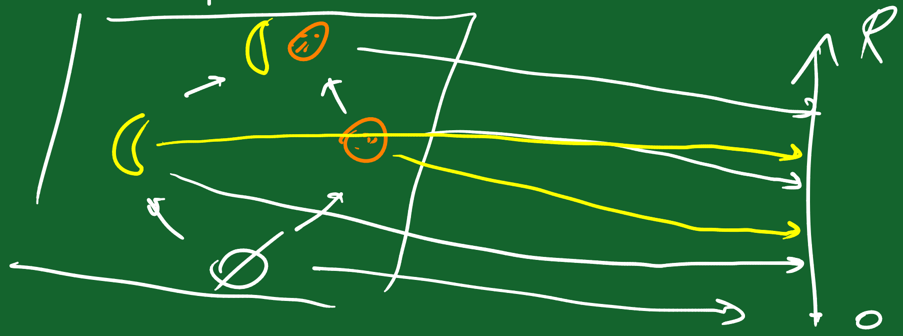
$\rightarrow$ Topological case!



→ 1) Compact Hausdorff spaces, closed partial order, continuous monotone maps

→ 2) Complete metric spaces, 1-Lipschitz monotone maps
+
order s.t.

$$\left(x \leq y \iff \forall f : X \to \mathbb{R},\ f(x) \leq f(y) \right)$$



Theorem: Let $p, q \in PX$

1) $p \leq q$ in the stoch. order $\iff$ $\forall$ cont. mon. map $X \to \mathbb{R}$, $\int f\, dp \leq \int f\, dq$

2) $p \leq q$ in the stoch. order $\iff$ $\forall$ 1-Lip. mon. map $X \to \mathbb{R}$, $\int f\, dp \leq \int f\, dq$

$\forall$ consistent choice of price, p is cheaper than q.

We get prob. 2-monads on these ordered sets.

1) Comp. Haus ... Radon monad

2) Compl. met ... (ordered) Kantorovich monad.

Monads : deterministic $\rightsquigarrow$ random

order on X $\qquad$ order on PX

$$X \xrightarrow{\delta} PX$$

Probabilistic powerdomain (DCPOs)

$$\begin{tikzcd} PX \arrow[r, bend left, "f_*"] \arrow[r, bend right, "g_*"'] \arrow[r, phantom, "\leq"] & PY \end{tikzcd}$$

"P is monotone"

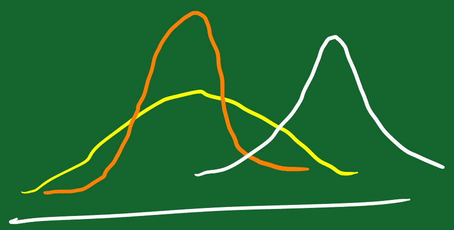
Algebras :

1) compact convex subsets of loc convex top. v. space with a closed pos. cone

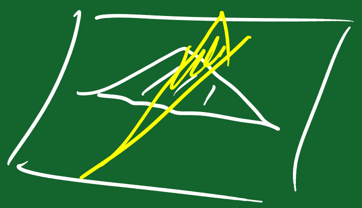
2) compact convex subsets of Banach space with a closed pos. cone

Second-order stoch. dominance : how random ? - on the same data.

$(\Omega, \mathcal{F}, \mu)$



X



X

Def. $\Omega \underset{g}{\overset{f}{\rightrightarrows}} \mathbb{R}$, any σ-algebra

g is a conditional expectation of f

if $\exists\ \mathcal{G} \subseteq \mathcal{F}$ s.t.

1) g is $\mathcal{G}$-measurable

2) $\forall\ G \in \mathcal{G}$, $\int_G f\, d\mu = \int_G g\, d\mu$

$\Rightarrow$ use partitions!

Let $p, q \in PA$.

TFAE.

1) $\exists$ $\Omega \underset{g}{\overset{f}{\rightrightarrows}} A$ $\quad f_*\mu = p, \; g_*\mu = q,\;$ g is a cond. exp. of f.

2) $\exists\, r \in PPA$ s.t.

$$PA \xleftarrow{E} PPA \xrightarrow{Pe} PA$$

$Er = p$

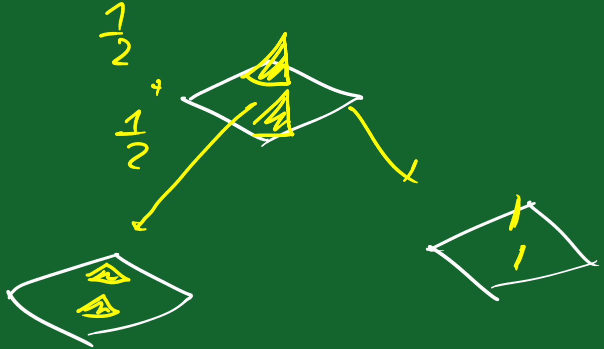
$Pe(r) = q$.

3) $\forall$ map $f : A \to \mathbb{R}$ concave, $\int f\, dp \leq \int f\, dq$.

2) A compl. m. space ...

P. alg.

$$PA \xrightarrow{e} A$$
$$a \longmapsto \int a \, dp(a)$$



oplax morph. of algebras.

References :

1) K. Keimel, "The monad of p. measures over comp. ord. spaces", '08

2) www.paoloperrone.org/phdthesis.pdf.

(Chapter 4)