Quantum logic is undecidable

arXiv:1607.05870

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What is quantum logic?

- Idea: Quantum weirdness is an illusion due to reasoning in Boolean logic, which is inadequate at the quantum level.
- Example: in Boolean logic, $\wedge$ distributes over $\vee$,

$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R),$$

  where P, Q and R propositions. Not so in quantum logic!
- Quantum propositions are projection operators on Hilbert space, or equivalently closed subspaces.
- Connectives of quantum logic:
  - $\wedge$ is intersection of subspaces,
  - $\vee$ is the closed linear span,
  - Negation is the orthogonal complement.
- Example where the above distributivity fails?

The laws of quantum logic

So which laws of logic are valid quantumly?

- Some rules of Boolean logic still apply, e.g.

$$P \vee P^{\perp} = 1, \qquad P \wedge P^{\perp} = 0.$$

- Orthomodularity: if $P \leq Q$, then

$$P \vee (P^{\perp} \wedge Q) = Q.$$

- These are some particular laws. Is it possible to classify all of them?
- More precise question: what is the **complexity** of telling whether a given candidate law is valid or not?

Complexity of quantum logic

Theorem

There is no algorithm to decide whether an implication of the form

$$(E_1 \text{ and } E_2 \text{ and } \ldots \text{ and } E_k) \text{ implies } F$$

holds for all Hilbert spaces, where each E_i as well as F has one of the following two forms:

- an equation phrased solely in terms of free variables, lattice join $\vee$, and 0;
- an orthogonality relation $\perp$ between two free variables.

Example:

- $P \vee Q = 1$ and $Q \vee R = 1$ and $R \vee P = 1$ and all pairwise orthogonalities implies $P = 0$.

Our proof shows undecidability for an even more specific class of implications, as follows.

Lemma

For projections $P_1, \ldots, P_n$, the following are equivalent:

- $\sum_i P_i = 1$,
- $P_1 \vee \ldots \vee P_n = 1$ and $P_i \perp P_j$ for all i, j.

We denote this by $OC(P_1, \ldots, P_n)$. It is a conjunction of premises of the above form.

Definition

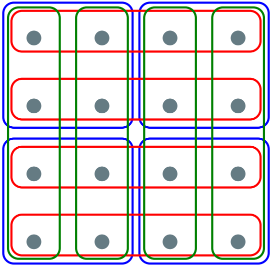
A **hypergraph** (V, E) is a finite set V together with a collection of subsets $E \subseteq 2^V$ called **hyperedges**.

For families of projections $(P_v)_{v \in V}$ indexed by $v \in V$, we write $OC(\bar{P}_e)$ for the above condition applied to $(P_v)_{v \in e}$ for some $e \in E$.

Decision Problem[1]

Given a hypergraph (V, E), is there a **quantum representation** consisting of projections $(P_v)_{v \in V}$ such that $OC(\bar{P}_e)$ for all e?

For example:



Next example: same, but with some nodes removed!

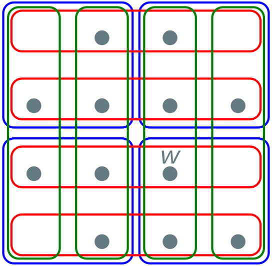
No algorithm is known, but proving undecidability seems hard.

[1] Antonio Acín, Tobias Fritz, Anthony Leverrier and Ana Belén Sainz, A Combinatorial Approach to Nonlocality and Contextuality, arXiv:1212.4084.

Decision Problem

Given a hypergraph (V, E) and $w \in V$, is $P_w = 0$ in every quantum representation of (V, E)?

Example:[2]



Our verdict is:

Main Theorem

This problem is undecidable.

[2] Tobias Fritz, Quantum analogues of Hardy's nonlocality paradox, arXiv:1006.2497.

Hypergraph C*-algebras

For the proof, we will use that quantum representations of (V, E) are the same thing as representations of the **hypergraph C*-algebra** $C^*(V, E)$,

$$C^*(V,E) = \left\langle P_v,\ v \in V \;\middle|\; P_v^2 = P_v = P_v^*, \quad \sum_{v \in e} P_v = 1 \right\rangle$$

The decision problem then asks whether $P_w = 0$ in $C^*(V, E)$.

Aside: The undecidability implies that

Theorem

Infinitely many of the C*-algebras $C^*(V, E)$ are not residually finite-dimensional.

This is the first time that the connection[3] between computability and residual finite-dimensionality has been successfully applied.

[3] Tobias Fritz, Tim Netzer, Andreas Thom, Can you compute the operator norm?, arXiv:1207.0975.

Another kind of algebraic structure plays a role in the proof:

Definition (Cleve, Liu, Slofstra[4] with minor modification)

The **solution group** associated to a bipartite graph $G = I \cup T$ is the group with generators $(x_i)_{i \in I}$ and relations

- $x_i^2 = 1$ for all $i \in I$,
- $x_i x_j = x_i x_j$ for all i, j with $i, j \sim t$ for some $t \in T$,
- $\prod_{i \in t} x_i = 1$ for all $t \in T$.

Decision Problem

Given (V, E) and $w \in V$, is $x_w = 1$ in the solution group $\Gamma(V, E)$?

Theorem (Slofstra[5])

This problem is undecidable.

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[4] Richard Cleve, Li Liu, William Slofstra, Perfect Commuting-Operator Strategies for Linear System Games, arXiv:1606.02278.

[5] William Slofstra, Tsirelson's problem and an embedding theorem for groups arising from non-local games, arXiv:1606.03140.

We now leverage Slofstra's result to prove our main theorem.

Lemma

A solution group C*-algebra $C^*(G)$ is computably isomorphic to a hypergraph C*-algebra $C^*(V, E)$ for a suitable (V, E).

Idea of proof:

- The x_i are ± 1-valued projective measurements.
- For every $t \in T$ there is a measurement corresponding to joint measurement of the $\{x_i : i \sim t\}$;
- The outcomes for which the parity of such a measurement is -1 are removed.
- This results in a contextuality scenario described by a hypergraph.

The isomorphism is such that $x_w = 1$ if and only if $P_w = 0$.

The role of Hilbert space dimension

Final Remarks

- The undecidability relies crucially on the infinite-dimensionality on Hilbert space!
- The analogous decision problem in a fixed range of dimensions is decidable thanks to real quantifier elimination.
- For arbitrary finite Hilbert space dimension, undecidability of quantum logic was already known[6].

[6] Leonard Lipshitz, The Undecidability of the Word Problems for Projective Geometries and Modular Lattices, jstor.org/stable/1996907.