The Inflation Technique for Causal Inference with Latent Variables

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Introduction

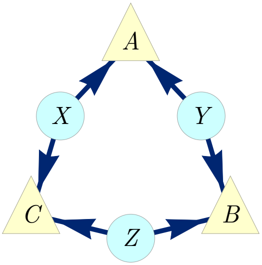
Given some correlations between the vocabulary of three languages, under what conditions

- does it follow that **all three** must have a common ancestor,

as opposed to

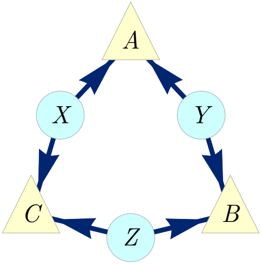
- at most every two of them can have a common ancestor?

Diagrammatically:



An example

More generally: given a joint distribution P_{ABC}, can it arise from the Triangle graph?

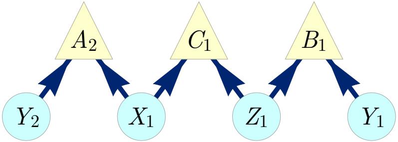


Example

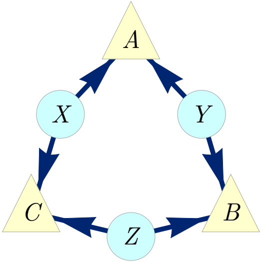
Binary variables with perfect correlation:

$$P_{ABC} = \frac{[000] + [111]}{2}$$

To solve the example problem, let's consider a slightly different graph:



Assuming that the causal dependences are as in the original



some marginals coincide: $P_{A_2C_1} = P_{AC}$ and $P_{C_1B_1} = P_{CB}$.

But: A_2 and C_1 are independent, $P_{A_2C_1} = P_A P_C$.

Hence A_2 and C_1 are perfectly correlated, as are C_1 and B_1, while A_2 and B_1 are independent.

There is no joint distribution with these properties!
$\Rightarrow P_{ABC}$ is incompatible with the Triangle graph.

We can make this quantitative by deriving a **causal compatibility inequality**. The existence of a joint distribution implies a constraint on marginal distributions,

$$\langle A_2 C_1 \rangle + \langle C_1 B_1 \rangle \leq 1 + \langle A_2 B_1 \rangle.$$

Write the marginals in terms of the distribution on the Triangle to show:

Theorem

Every P_{ABC} with $\{\pm 1\}$-valued variables compatible with the Triangle graph satisfies

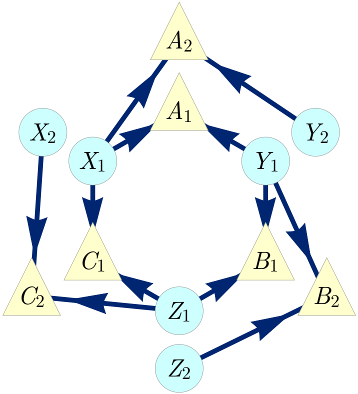
$$\langle AC \rangle + \langle BC \rangle \leq 1 + \langle A \rangle \langle B \rangle.$$

Another example

Is

$$P_{ABC} = \frac{[001] + [010] + [100]}{2}$$

compatible with the Triangle graph? Let's consider the **Spiral inflation graph**, where having the same causal dependences implies:



$$
\begin{aligned}
P_{A_1B_1C_1} &= P_{ABC} \\
P_{A_1B_2C_2} &= P_{AB}P_C \\
P_{A_2B_1C_2} &= P_{BC}P_A \\
P_{A_2B_2C_1} &= P_{AC}P_B \\
P_{A_2B_2C_2} &= P_A P_B P_C.
\end{aligned}
$$

These marginals are such that $A_2 = B_2 = C_2 = 1$ has positive probability.

Whenever this event happens, also one of the following must happen:

- $A_1 = B_2 = C_2 = 1$,
- $A_2 = B_1 = C_2 = 1$,
- $A_2 = B_2 = C_1 = 1$,
- $A_1 = B_1 = C_1 = 0$.

However, all of these have probability zero!

$\Rightarrow$ There is no joint distribution for all six variables that reproduces these marginals.

$\Rightarrow$ The original distribution P_{ABC} is not compatible with the Triangle graph.

Again one can make this inference quantitative by deriving an inequality.

At the level of the Spiral inflation, the union bound implies that

$$\begin{aligned} P_{A_2B_2C_2}(111) \leq P_{A_1B_2C_2}(111) + P_{A_2B_1C_2}(111) \\ + P_{A_2B_2C_1}(111) + P_{A_1B_1C_1}(000) \end{aligned}$$

in every joint distribution.

The above equations for the marginals translate this into

$$\begin{aligned} P_A(1)P_B(1)P_C(1) \leq P_{AB}(11)P_C(1) + P_{BC}(11)P_A(1) \\ + P_{AC}(11)P_B(1) + P_{ABC}(000), \end{aligned}$$

which is another causal compatibility inequality for the Triangle graph.

The inflation technique

So what is the general method?

Definition

Given a graph G, an **inflation graph** is a graph G' together with a graph map $\pi : G' \to G$ such that its restriction to the ancestry of any node is a bijection.

An equivalent condition is that $G' \to G$ must be a **fibration**, i.e. that every edge in G with a lift of its target to G' lifts uniquely to G'.

In our figures, we specify the map by labelling each node in G' by the label of its image in G and a "copy index".

Every causal model on G **inflates** to a causal model on G'.

Definition

A set of nodes $\mathbf{U} \subseteq G'$ is **injectable** if $\pi|_{\mathbf{U}}$ is bijective.

The distribution on an injectable set in an inflation model is specified by the corresponding marginal distribution on G.

Lemma

If a distribution on observable nodes of G is compatible with G', then the associated family of distributions on injectable sets is compatible with G'.

This is the central observation that makes the inflation technique work.

Thus we can apply any method for causal inference on G', and the inflation technique translates it to causal inference on G.

This can amplify the power of causal inference methods significantly. In the earlier examples, we have only used two things at the level of G',

- The existence of a joint distribution,
- Sets of nodes with disjoint ancestry are independent.

For G, we obtain inequalities that do not just follow from the same requirements at the level of G!

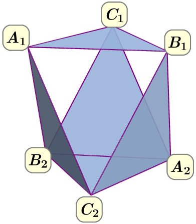
The sets that are relevant for applying the inflation technique in conjuction with the above two constraints on G' are:

Definition

A set of nodes $\mathbf{U} \subseteq G'$ is **pre-injectable** if it is the union of injectable sets with disjoint ancestry.

The marginal problem

When applying the inflation technique like this, we need to figure out when the resulting family of marginal distributions on pre-injectable sets can arise from a joint distribution.



This is the **marginal problem**.

The families of distributions that arise in this way form the **marginal polytope**. Its facet inequalities are what we turn into causal compatibility inequalities for G.

Thus the computational problems are those of **linear programming** and **facet enumeration** for the marginal polytope.

Results

Facet enumeration for the marginal polytope of the Spiral inflation with binary variables results in 37 symmetry classes of nontrivial inequalities:

$(\sharp 1^{*})$: $0 \le 1 + \langle B\rangle \langle C\rangle + \langle AB\rangle + \langle AC\rangle$

$(\sharp 2)$: $0 \le 2 + \langle A\rangle \langle B\rangle \langle C\rangle - \langle C\rangle \langle AB\rangle - 2\langle AC\rangle$

$(\sharp 3^{*})$: $0 \le 3 + \langle A\rangle - \langle B\rangle + \langle C\rangle + \langle B\rangle \langle C\rangle + \langle A\rangle \langle B\rangle \langle C\rangle + \langle AB\rangle - \langle C\rangle \langle AB\rangle + 3\langle AC\rangle - \langle B\rangle \langle AC\rangle$

$(\sharp 4^{*})$: $0 \le 3 + \langle A\rangle - \langle B\rangle + \langle C\rangle + \langle B\rangle \langle C\rangle - \langle A\rangle \langle B\rangle \langle C\rangle + \langle AB\rangle + \langle C\rangle \langle AB\rangle + 3\langle AC\rangle - \langle B\rangle \langle AC\rangle$

$(\sharp 5)$: $0 \le 3 + \langle B\rangle - \langle A\rangle \langle B\rangle + \langle A\rangle \langle C\rangle + \langle A\rangle \langle B\rangle \langle C\rangle + \langle AB\rangle + \langle C\rangle \langle AB\rangle - \langle B\rangle \langle AC\rangle - 2\langle BC\rangle$

$(\sharp 6)$: $0 \le 3 + \langle B\rangle + \langle A\rangle \langle C\rangle + \langle A\rangle \langle B\rangle \langle C\rangle - \langle C\rangle \langle AB\rangle - 2\langle AC\rangle - \langle B\rangle \langle AC\rangle - 2\langle BC\rangle$

$(\sharp 7)$: $0 \le 3 + \langle B\rangle + \langle A\rangle \langle B\rangle - \langle A\rangle \langle C\rangle - \langle A\rangle \langle B\rangle \langle C\rangle + \langle AB\rangle + \langle C\rangle \langle AB\rangle + \langle B\rangle \langle AC\rangle - 2\langle BC\rangle$

$(\sharp 8^{*})$: $0 \le 3 + \langle A\rangle + \langle B\rangle + \langle A\rangle \langle B\rangle + \langle C\rangle + \langle A\rangle \langle C\rangle + \langle B\rangle \langle C\rangle - \langle A\rangle \langle B\rangle \langle C\rangle + 2\langle AB\rangle + \langle C\rangle \langle AB\rangle + 2\langle AC\rangle + \langle B\rangle \langle AC\rangle + 2\langle BC\rangle + \langle A\rangle \langle BC\rangle - \langle ABC\rangle$

$(\sharp 9^{*})$: $0 \le 3 + \langle A\rangle + \langle B\rangle - \langle A\rangle \langle B\rangle + \langle C\rangle + \langle A\rangle \langle C\rangle + \langle B\rangle \langle C\rangle - \langle A\rangle \langle B\rangle \langle C\rangle + \langle C\rangle \langle AB\rangle + 2\langle AC\rangle + \langle B\rangle \langle AC\rangle - 2\langle BC\rangle - \langle A\rangle \langle BC\rangle + \langle ABC\rangle$

$(\sharp 10^{*})$: $0 \le 4 + 2\langle A\rangle \langle B\rangle + 2\langle C\rangle + 2\langle B\rangle \langle C\rangle - 2\langle AB\rangle + \langle C\rangle \langle AB\rangle - 2\langle AC\rangle + \langle B\rangle \langle AC\rangle + \langle A\rangle \langle BC\rangle - \langle ABC\rangle$

$(\sharp 11^{*})$: $0 \le 4 - 2\langle B\rangle + \langle B\rangle \langle C\rangle + \langle A\rangle \langle B\rangle \langle C\rangle - 2\langle AB\rangle + \langle C\rangle \langle AB\rangle - \langle B\rangle \langle AC\rangle - 3\langle BC\rangle + \langle ABC\rangle$

$(\sharp 12^{*})$: $0 \le 4 - 2\langle B\rangle + 2\langle A\rangle \langle C\rangle + \langle B\rangle \langle C\rangle - \langle A\rangle \langle B\rangle \langle C\rangle - 2\langle AB\rangle + \langle C\rangle \langle AB\rangle - 2\langle AC\rangle + \langle B\rangle \langle AC\rangle - 3\langle BC\rangle + \langle ABC\rangle$

$(\sharp 13^{*})$: $0 \le 4 - 2\langle A\rangle \langle B\rangle - 2\langle A\rangle \langle C\rangle - \langle B\rangle \langle C\rangle - \langle A\rangle \langle B\rangle \langle C\rangle + 2\langle AB\rangle - \langle C\rangle \langle AB\rangle - 2\langle AC\rangle + \langle B\rangle \langle AC\rangle + \langle BC\rangle + \langle ABC\rangle$

$(\sharp 14^{*})$: $0 \le 4 - 2\langle A\rangle \langle B\rangle + 2\langle A\rangle \langle C\rangle - \langle B\rangle \langle C\rangle + \langle A\rangle \langle B\rangle \langle C\rangle + 2\langle AB\rangle + \langle C\rangle \langle AB\rangle - 2\langle AC\rangle + \langle B\rangle \langle AC\rangle + \langle BC\rangle + \langle ABC\rangle$

$(\sharp 15^{*})$: $0 \le 4 + 2\langle A\rangle \langle C\rangle + \langle B\rangle \langle C\rangle + \langle A\rangle \langle B\rangle \langle C\rangle - \langle C\rangle \langle AB\rangle - 2\langle AC\rangle - \langle B\rangle \langle AC\rangle + 3\langle BC\rangle + \langle ABC\rangle$

$(\sharp 16^{*})$: $0 \le 4 - 2\langle B\rangle + 2\langle A\rangle \langle C\rangle - 2\langle AB\rangle + \langle C\rangle \langle AB\rangle - 2\langle AC\rangle + \langle B\rangle \langle AC\rangle - 2\langle BC\rangle - \langle A\rangle \langle BC\rangle + \langle ABC\rangle$

$(\sharp 17^{*})$: $0 \le 4 + 2\langle A\rangle \langle B\rangle + 2\langle A\rangle \langle C\rangle + 2\langle B\rangle \langle C\rangle - 2\langle AB\rangle + \langle C\rangle \langle AB\rangle - 2\langle AC\rangle + \langle B\rangle \langle AC\rangle - 2\langle BC\rangle + \langle A\rangle \langle BC\rangle + \langle ABC\rangle$

$(\sharp 18)$: $0 \le 5 + \langle A\rangle + \langle B\rangle - 2\langle A\rangle \langle B\rangle + \langle C\rangle + \langle B\rangle \langle C\rangle + \langle A\rangle \langle B\rangle \langle C\rangle + 3\langle AB\rangle + \langle C\rangle \langle AB\rangle + \langle AC\rangle - \langle B\rangle \langle AC\rangle - 4\langle BC\rangle$

$(\sharp 19^{*})$: $0 \le 5 + \langle A\rangle + \langle B\rangle + 2\langle A\rangle \langle B\rangle + \langle C\rangle - 2\langle A\rangle \langle C\rangle + \langle B\rangle \langle C\rangle - \langle A\rangle \langle B\rangle \langle C\rangle + 3\langle AB\rangle + \langle C\rangle \langle AB\rangle - \langle AC\rangle + \langle B\rangle \langle AC\rangle - 4\langle BC\rangle$

$(\sharp 20^{*})$: $0 \le 5 + \langle A\rangle - \langle B\rangle - 2\langle A\rangle \langle B\rangle + \langle C\rangle - \langle A\rangle \langle C\rangle + \langle B\rangle \langle C\rangle + \langle AB\rangle + \langle C\rangle \langle AB\rangle + 2\langle AC\rangle - 2\langle B\rangle \langle AC\rangle - 2\langle BC\rangle - 2\langle A\rangle \langle BC\rangle - 2\langle ABC\rangle$

$(\sharp 21^{*})$: $0 \le 5 + \langle A\rangle + \langle B\rangle + \langle C\rangle - \langle A\rangle \langle C\rangle - \langle B\rangle \langle C\rangle - 2\langle A\rangle \langle B\rangle \langle C\rangle + \langle AB\rangle + 2\langle C\rangle \langle AB\rangle + 2\langle AC\rangle + \langle B\rangle \langle AC\rangle - 2\langle BC\rangle + \langle A\rangle \langle BC\rangle - \langle ABC\rangle$

$(\sharp 22^{*})$: $0 \le 5 - \langle A\rangle + \langle B\rangle - 2\langle A\rangle \langle B\rangle + \langle C\rangle - 2\langle A\rangle \langle C\rangle + 2\langle B\rangle \langle C\rangle + \langle AB\rangle - 2\langle C\rangle \langle AB\rangle + \langle AC\rangle - 2\langle B\rangle \langle AC\rangle - \langle BC\rangle - 2\langle A\rangle \langle BC\rangle + \langle ABC\rangle$

$(\sharp 23^{*})$: $0 \le 5 + \langle A\rangle + \langle B\rangle - \langle A\rangle \langle B\rangle + \langle C\rangle + 2\langle B\rangle \langle C\rangle + \langle A\rangle \langle B\rangle \langle C\rangle + 2\langle AB\rangle - \langle C\rangle \langle AB\rangle + \langle AC\rangle - 2\langle B\rangle \langle AC\rangle - \langle BC\rangle - 2\langle A\rangle \langle BC\rangle + \langle ABC\rangle$

$(\sharp 24^{*})$: $0 \le 5 + \langle A\rangle + \langle B\rangle - 2\langle A\rangle \langle B\rangle + \langle C\rangle - \langle A\rangle \langle C\rangle - \langle B\rangle \langle C\rangle - 2\langle A\rangle \langle B\rangle \langle C\rangle - \langle AB\rangle + 2\langle C\rangle \langle AB\rangle + 2\langle AC\rangle + \langle B\rangle \langle AC\rangle + 2\langle BC\rangle - \langle A\rangle \langle BC\rangle + \langle ABC\rangle$

$(\sharp 25)$: $0 \le 6 + 2\langle A\rangle \langle B\rangle + \langle A\rangle \langle C\rangle + 2\langle B\rangle \langle C\rangle + \langle A\rangle \langle B\rangle \langle C\rangle - 4\langle AB\rangle - 2\langle C\rangle \langle AB\rangle - 3\langle AC\rangle - \langle B\rangle \langle AC\rangle - 2\langle A\rangle \langle BC\rangle$

$(\sharp 26)$: $0 \le 6 - 2\langle A\rangle + \langle A\rangle \langle B\rangle + 2\langle C\rangle + \langle A\rangle \langle C\rangle + 2\langle A\rangle \langle B\rangle \langle C\rangle - 5\langle AB\rangle - \langle C\rangle \langle AB\rangle - 3\langle AC\rangle + \langle B\rangle \langle AC\rangle - 2\langle A\rangle \langle BC\rangle$

$(\sharp 27)$: $0 \le 6 + 2\langle A\rangle \langle B\rangle + 2\langle C\rangle + \langle A\rangle \langle C\rangle + \langle A\rangle \langle B\rangle \langle C\rangle - 4\langle AB\rangle - 2\langle C\rangle \langle AB\rangle + 3\langle AC\rangle + \langle B\rangle \langle AC\rangle - 2\langle A\rangle \langle BC\rangle$

$(\sharp 28)$: $0 \le 6 + \langle A\rangle \langle B\rangle + \langle A\rangle \langle C\rangle - 4\langle B\rangle \langle C\rangle - 2\langle A\rangle \langle B\rangle \langle C\rangle + \langle AB\rangle + \langle C\rangle \langle AB\rangle - 3\langle AC\rangle - \langle B\rangle \langle AC\rangle + 2\langle BC\rangle - 2\langle A\rangle \langle BC\rangle$

$(\sharp 29)$: $0 \le 6 + 2\langle B\rangle + \langle A\rangle \langle B\rangle - 2\langle A\rangle \langle C\rangle + \langle B\rangle \langle C\rangle - 2\langle A\rangle \langle B\rangle \langle C\rangle + 3\langle AB\rangle + \langle C\rangle \langle AB\rangle + 2\langle B\rangle \langle AC\rangle - 5\langle BC\rangle + \langle A\rangle \langle BC\rangle$

$(\sharp 30)$: $0 \le 6 + 2\langle B\rangle - 2\langle A\rangle \langle B\rangle + 4\langle A\rangle \langle C\rangle - \langle B\rangle \langle C\rangle + \langle A\rangle \langle B\rangle \langle C\rangle + 2\langle AB\rangle + 2\langle C\rangle \langle AB\rangle - 2\langle AC\rangle + 2\langle B\rangle \langle AC\rangle + \langle BC\rangle + \langle A\rangle \langle BC\rangle$

$(\sharp 31)$: $0 \le 6 + \langle A\rangle \langle C\rangle + 4\langle B\rangle \langle C\rangle + \langle A\rangle \langle B\rangle \langle C\rangle - 2\langle AB\rangle - 2\langle C\rangle \langle AB\rangle - 3\langle AC\rangle - \langle B\rangle \langle AC\rangle - 2\langle BC\rangle - 2\langle ABC\rangle$

$(\sharp 32)$: $0 \le 7 + \langle A\rangle + \langle B\rangle + \langle A\rangle \langle B\rangle + \langle C\rangle - 2\langle A\rangle \langle C\rangle + 2\langle B\rangle \langle C\rangle - \langle A\rangle \langle B\rangle \langle C\rangle + 2\langle AB\rangle + 3\langle C\rangle \langle AB\rangle + \langle AC\rangle + 2\langle B\rangle \langle AC\rangle - 3\langle BC\rangle - 2\langle A\rangle \langle BC\rangle + 3\langle ABC\rangle$

$(\sharp 33)$: $0 \le 8 + 2\langle A\rangle \langle B\rangle + 4\langle A\rangle \langle C\rangle - 2\langle B\rangle \langle C\rangle + 2\langle A\rangle \langle B\rangle \langle C\rangle - 2\langle AB\rangle - \langle C\rangle \langle AB\rangle - 4\langle AC\rangle + \langle B\rangle \langle AC\rangle - 2\langle BC\rangle - 3\langle A\rangle \langle BC\rangle - 3\langle ABC\rangle$

$(\sharp 34)$: $0 \le 8 + 2\langle A\rangle - 2\langle C\rangle - \langle A\rangle \langle C\rangle + 2\langle B\rangle \langle C\rangle + 3\langle A\rangle \langle B\rangle \langle C\rangle - 6\langle AB\rangle + \langle C\rangle \langle AB\rangle + \langle AC\rangle + 2\langle B\rangle \langle AC\rangle - 3\langle A\rangle \langle BC\rangle + \langle ABC\rangle$

$(\sharp 35)$: $0 \le 8 + 2\langle A\rangle + \langle A\rangle \langle C\rangle + 2\langle B\rangle \langle C\rangle + 3\langle A\rangle \langle B\rangle \langle C\rangle + 6\langle AB\rangle - \langle C\rangle \langle AB\rangle + \langle AC\rangle - 2\langle B\rangle \langle AC\rangle - 2\langle BC\rangle - 3\langle A\rangle \langle BC\rangle + \langle ABC\rangle$

$(\sharp 36)$: $0 \le 8 - 2\langle B\rangle + 2\langle A\rangle \langle B\rangle - 2\langle C\rangle - \langle B\rangle \langle C\rangle - 3\langle A\rangle \langle B\rangle \langle C\rangle + 3\langle C\rangle \langle AB\rangle - 6\langle AC\rangle + \langle B\rangle \langle AC\rangle + \langle BC\rangle - 2\langle A\rangle \langle BC\rangle + \langle ABC\rangle$

$(\sharp 37)$: $0 \le 8 + 2\langle B\rangle + \langle A\rangle \langle B\rangle - 2\langle A\rangle \langle C\rangle - 3\langle A\rangle \langle B\rangle \langle C\rangle + \langle AB\rangle + 2\langle C\rangle \langle AB\rangle + 2\langle AC\rangle + 3\langle B\rangle \langle AC\rangle - 6\langle BC\rangle - \langle A\rangle \langle BC\rangle + \langle ABC\rangle$

Prospects

- Derive tighter constraints by using more than just ancestral independences on the inflation graph?

  Possibilities:

  - conditional independences,
  - coinciding distributions on isomorphic subgraphs.

- Necessary **and sufficient** constraints?

  $\rightarrow$ Considering larger and larger inflation graphs should allow for reconstruction of the dependences on the hidden variables.

- Investigate the link to entropic inequalities!

Entropic inequalities

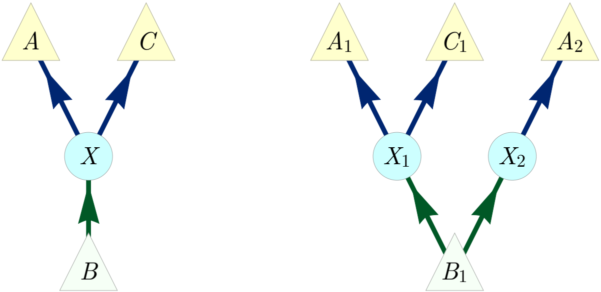
The **laws of information theory** (Yeung):

- Submodularity, $H(AB) + H(BC) \geq H(B) + H(ABC)$.
- **Non-Shannon-type** inequalities, such as the Zhang–Yeung inequality,

  $$\begin{aligned} &3H(AC) + 3H(AD) + H(BC) + H(BD) + 3H(CD) \\ &\qquad \geq 4H(ACD) + H(BCD) + H(AB) + H(A) + 2H(C) + 2H(D), \end{aligned}$$

  which are not consequences of submodularity.
- These are the inequalities that bound the entropy cone.
- Finding a complete list of non-Shannon-type inequalities is an open problem.

The derivation of the known non-Shannon-type inequalities relies on the **copy lemma**, which secretly is an application of the inflation technique:



Problem

Is it possible to derive new non-Shannon-type inequalities by considering other inflation graphs?