Almost C*-algebras

November 2015

Appetizer: a problem in point-set topology

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- So far, we have been able to show this with the unit square $[0,1]^2$ in place of $[0,1]$.
- This is a crucial ingredient in the technical development of almost C*-algebras.

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- In the following, all C*-algebras will be assumed unital.

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- These theorems also showcase the fundamental examples of C*-algebras: $C(X)$ and $\mathcal{B}(\mathcal{H})$.

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- However, this is very challenging: not even the physical meaning of the multiplication is clear!
- We try to improve on this by attempting to reaxiomatize C*-algebras. Currently only partial results.

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- We would like this map to be injective: every two measurements can be combined to a joint measurement in at most one way.

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- We furthermore require certain **sheaf conditions**, such as: with $\bigcirc$ the unit disk, the $\alpha \in M(X)$ should be in bijection with families $(\beta_f)_{f : X \to \bigcirc}$ where $\beta_f \in M(\bigcirc)$ and

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- If M satisfies the sheaf conditions, we call it a **sheaf**.

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- Example: the normal elements of any C*-algebra form a piecewise C*-algebra.

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- Hence we axiomatize the action of inner automorphisms as an extra piece of structure.

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- Every C*-algebra carries the structure of an almost C*-algebra.

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If A is a von Neumann algebra, then every almost $*$-homomorphism $A \to B$ is a $*$-homomorphism.

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