A Combinatorial Approach to Nonlocality and Contextuality

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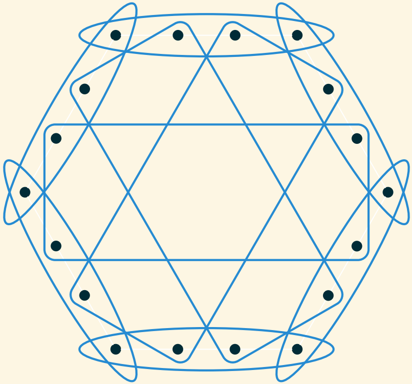
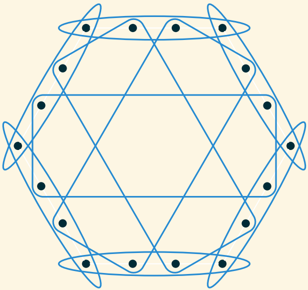
Cabello's proof of the Kochen-Specker theorem



- 18 vectors in $\mathbb{C}^4$ corresponding to the vertices,
- 9 bases (=projective measurements) corresponding to blue edges,
- There is no way to assign 0's and 1's to the vertices such that there is exactly one 1 in each edge. $\Rightarrow$ Contextuality!

Contextuality scenarios



General setup: a **contextuality scenario** is a hypergraph H with

- a set of vertices $V(H)$ representing measurement outcomes,
- a set of hyperedges $E(H)$ representing measurements,
- different measurements may have outcomes in common. (See Spekkens' measurement noncontextuality).

Same as a test space!

Probabilistic models

Definition

A **probabilistic model** on H is an assignment of a probability $p(v)$ to each outcome v such that probabilities are normalized:

$$\sum_{v \in e} p(v) = 1,$$

for each measurement $e \in E(H)$.

- Same as a **state** on a test space!
- The set of probabilistic models is denoted $\mathcal{G}(H)$.

Quantum models

Definition

A **quantum model** on H is an assignment of probabilities $v \mapsto p(v)$ for which there exist a Hilbert space $\mathcal{H}$ together with a state $|\psi\rangle \in \mathcal{H}$ and an assignment of **projections** $v \mapsto P_v$ on $\mathcal{H}$ such that the projections are normalized,

$$\sum_{v \in e} P_v = \mathbb{1},$$

and the probabilities are recovered, $p(v) = \langle\psi|P_v|\psi\rangle$.

- Every quantum model is a probabilistic model:

$$\sum_{v \in e} p(v) = \sum_{v \in e} \langle\psi|P_v|\psi\rangle = \left\langle \psi \middle| \sum_{v \in e} P_v \middle| \psi \right\rangle = \langle\psi|\psi\rangle = 1.$$

- The set of all quantum models is denoted $\mathcal{Q}(H)$.

Classical models

Definition

1. A **deterministic model** is an assignment of 0 or 1 to every outcome such that there is exactly one 1 in each measurement.
2. A **classical model** is an assigment of probabilities $v \mapsto p(v)$ which is a convex combination of deterministic models.

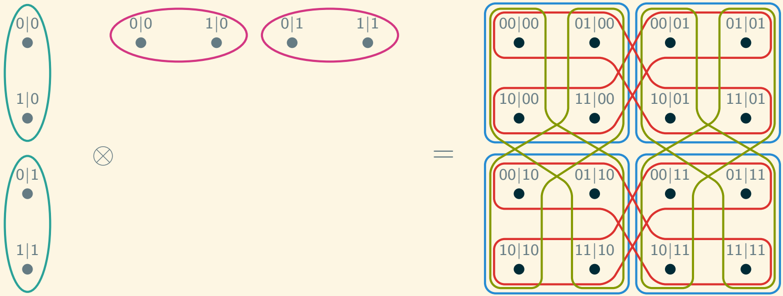
- Every classical model is quantum.
- The classical models are exactly those which can be obtained with a noncontextual deterministic hidden variable.
- The set of all classical models is denoted $\mathcal{C}(H)$.
- Some scenarios H have quantum models, but no classical models $\Rightarrow$ proof of the Kochen-Specker theorem!

Products and Bell scenarios

- If Alice operates in a scenario H_A and Bob in H_B, their joint measurements live in a scenario $H_A \otimes H_B$.
- The definition of $H_A \otimes H_B$ coincides with the **Foulis-Randall product** of test spaces.
- Operationally, the measurements in $H_A \otimes H_B$ are of three kinds:
  1. a pair of independently conducted measurements,
  2. joint measurements in which Alice measurements first, communicates her outcome to Bob, who then chooses his measurement as a function of Alice's outcome,
  3. joint measurements in which Alice's measurement is likewise a function of Bob's outcome.
- The latter two kinds of joint measurements enforce that every probabilistic model on $H_A \otimes H_B$ is no-signalling.

Products and Bell scenarios

- Example:



- This is the CHSH scenario!
- In this one and in any other Bell scenario, we have:
  - probabilistic model = no-signaling box,
  - quantum model = quantum correlation,
  - classical model = local correlation.

Consistent Exclusivity, level 1

Which properties distinguish quantum models from all the other probabilistic models? One possible answer is this:

Definition

A probabilistic model p satisfies **Consistent Exclusivity** if for every set of pairwise compatible outcomes $C \subseteq V$,

$$\sum_{v \in C} p(v) \leq 1,$$

where a pair of outcome is compatible if they are outcomes of the same measurement.

- Quantum models satisfy Consistent Exclusivity, because: if $\{P_v\}_{v \in C}$ is a family of pairwise orthogonal projections, then $\sum_{v \in C} P_v \leq \mathbb{1}$.
- The set of all probabilistic models satisfying Consistent Exclusivity is denoted $\mathcal{CE}^1(H)$.
- In Bell scenarios: "Local Orthogonality"

Consistent Exclusivity, level ∞

- Consistent Exclusivity can be **activated**: there are probabilistic models $p \in \mathcal{CE}^1(H)$ such that $p \otimes p \notin \mathcal{CE}^1(H \otimes H)$.
- This happens e.g. for $p =$ the PR-box.

Definition

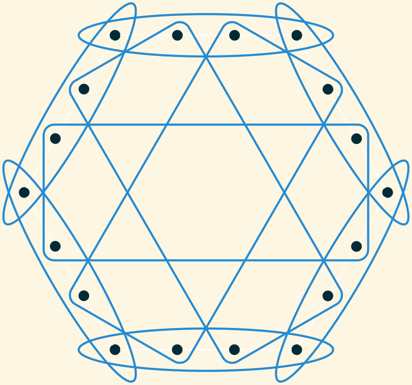
p satisfies **Consistent Exclusivity at level** ∞ if $p^{\otimes n}$ satisfies Consistent Exclusivity for each $n \in \mathbb{N}$.

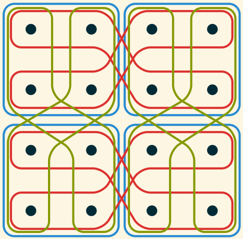
- The set of probabilistic models satisfying Consistent Exclusivity at level ∞ is denoted $\mathcal{CE}^\infty(H)$.

Relation to graph invariants

Inspired by "(Non-)Contextuality of Physical Theories as an Axiom".

Orthogonality graph $\mathrm{Ort}(H)$:





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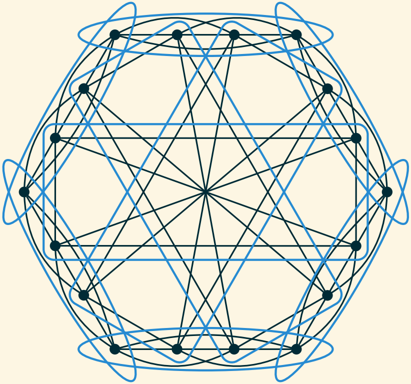

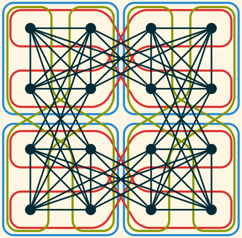

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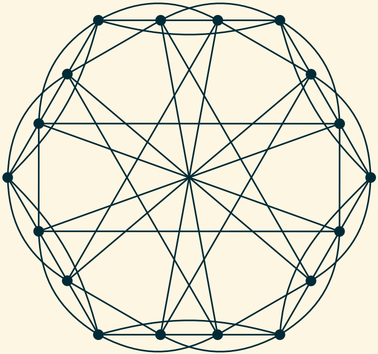

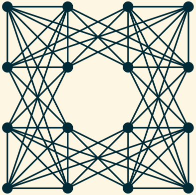

Relation to graph invariants

Non-orthogonality graph $\mathrm{NO}(H) := \overline{\mathrm{Ort}(H)}$ equipped with vertex weights p.

$p \in \mathcal{C}(H)$	$\Leftrightarrow$	$\alpha^*(\mathrm{NO}(H), p) = 1$	(fractional packing number)
$p \in \mathcal{Q}_1(H)$	$\Leftrightarrow$	$\vartheta(\mathrm{NO}(H), p) = 1$	(Lovász number)
$p \in \mathcal{CE}^\infty(H)$	$\Leftrightarrow$	$\Theta(\mathrm{NO}(H), p) = 1$	(Shannon capacity)
$p \in \mathcal{CE}^1(H)$	$\Leftrightarrow$	$\alpha(\mathrm{NO}(H), p) = 1$	(independence number)

- $\mathcal{Q}_1(H)$ is a certain relaxation of $\mathcal{Q}(H)$ defined in terms of a semidefinite program.
- The four sets on the left form an increasing sequence:

$$\mathcal{C} \subseteq \mathcal{Q}_1 \subseteq \mathcal{CE}^\infty \subseteq \mathcal{CE}^1.$$

  This is equivalent to well-known inequalities between graph invariants:

$$\alpha^* \geq \vartheta \geq \Theta \geq \alpha$$

Relation to graph invariants

- The quantum set $\mathcal{Q}(H)$ has not appeared in the previous list.
- So what about a graph invariant associated to $p \in \mathcal{Q}(H)$ itself?

Theorem

There are scenarios H and H' with $\mathrm{NO}(H) = \mathrm{NO}(H')$, together with a probabilistic model p on both H and H' such that

$$\boxed{p \in \mathcal{Q}(H), \qquad p \notin \mathcal{Q}(H').}$$

Corollary

$\mathcal{Q}(H)$ cannot be characterized in terms of a graph invariant of $\mathrm{NO}(H)$.

Non-convexity of $\mathcal{CE}^\infty$

- The relation to graph invariants lets us prove this:

Theorem

There are scenarios H, H_A and H_B such that

1. violations can be activated,

$$\mathcal{CE}^\infty(H_A) \otimes \mathcal{CE}^\infty(H_B) \not\subseteq \mathcal{CE}^\infty(H_A \otimes H_B),$$

2. non-convexity: $\mathcal{CE}^\infty(H)$ is not convex.

- Our explicit examples are quite big: H has 12 320 outcomes!

New results on the Shannon capacity of graphs

- Again by using the relation to graph invariants, we can turn the previous result into a theorem about these:

Theorem

There are graphs G_1 and G_2 having the following properties:

$$\boxed{\begin{array}{ll} \Theta(G_1) = \alpha(G_1) & \Theta(G_1 + G_2) > \Theta(G_1) + \Theta(G_2) \\ \vartheta(G_2) = \alpha(G_2) & \Theta(G_1 \boxtimes G_2) > \Theta(G_1) \cdot \Theta(G_2) \end{array}}$$

- This strengthens results of Haemers and Alon on counterexamples to questions of Lovász and Shannon.
- In our explicit example, G_1 has 220 vertices, while G_2 has 1 131 460!

An inverse sandwich conjecture

The sandwich theorem:

Theorem (Lovász)

- Lovász number, **easy** to compute

$$\alpha(G) \;\leq\; \vartheta(G) \;\leq\; \chi(\overline{G})$$

- Independence number, **hard** to compute
- Chromatic number, **hard** to compute

An inverse sandwich conjecture

Theorem + Conjecture

- Quantum set, membership **undecidable?**

$$\mathcal{C}(H) \subseteq \mathcal{Q}(H) \subseteq \mathcal{G}(H)$$

- Classical set, membership **decidable**
- General probabilistic set, membership **decidable**

- The first item is conjectural.
- The conjecture is that the meat lies in the middle of the sandwich!

Ramifications of a potential proof:

- an interesting class of new examples of C^*-algebras without a certain finite-dimensional approximation property,
- related to conjectural undecidability of quantum logic.

Further reading

- Main paper: Acín, Fritz, Leverrier, Sainz,
  - A Combinatorial Approach to Nonlocality and Contextuality
  - Probabilistic models on contextuality scenarios
- Contextuality and graph theory: Cabello, Severini, Winter,
  - (Non-)Contextuality of Physical Theories as an Axiom
  - Graph-Theoretic Approach to Quantum Correlations
- Local Orthogonality: Acín, Augusiak, Brask, Chaves, Fritz, Leverrier, Sainz,
  - Local orthogonality as a multipartite principle for quantum correlations
  - Exploring the Local Orthogonality Principle
- An observable-based approach to nonlocality and contextuality: Abramsky, Brandenburger,
  - The Sheaf-Theoretic Structure Of Non-Locality and Contextuality
- Operational aspects of contextuality: Spekkens,
  - Contextuality for preparations, transformations, and unsharp measurements
  - What is the appropriate notion of noncontextuality for unsharp measurements in quantum theory?