PMATH 911: Assignment 1

Solutions by Forte Shinko and Cecylia Bocovich, with minor modifications

Question 1

We will derive the following judgment:

$$A : \texttt{Type} \vdash \lambda f. f(\lambda x.x) \colon ((A \to A) \to A) \to A$$

First of all, we can derive:

$$\dfrac{\dfrac{\cdot\ \texttt{ctx}}{(A \colon \texttt{Type})\ \texttt{ctx}}}{A \colon \texttt{Type} \vdash A \colon \texttt{Type}}$$

Thus we can derive:

$$\dfrac{A \colon \texttt{Type} \vdash A \colon \texttt{Type} \qquad A \colon \texttt{Type} \vdash A \colon \texttt{Type}}{A \colon \texttt{Type} \vdash A \to A \colon \texttt{Type}}$$

Thus we can further derive:

$$\dfrac{\dfrac{A \colon \texttt{Type} \vdash A \to A \colon \texttt{Type} \qquad A \colon \texttt{Type} \vdash A \colon \texttt{Type}}{A \colon \texttt{Type} \vdash (A \to A) \to A \colon \texttt{Type}}}{(A \colon \texttt{Type}, f \colon (A \to A) \to A)\ \texttt{ctx}}$$

Now we'll split up into two directions.
The easy branch is this one:

$$\dfrac{(A \colon \texttt{Type}, f \colon (A \to A) \to A)\ \texttt{ctx}}{A \colon \texttt{Type}, f \colon (A \to A) \to A \vdash f \colon (A \to A) \to A}$$

The other one is the derivation of the identity map:

$$\dfrac{\dfrac{\dfrac{A \colon \texttt{Type}, f \colon (A \to A) \to A) \vdash A \colon \texttt{Type}}{(A \colon \texttt{Type}, f \colon (A \to A) \to A, x \colon A)\ \texttt{ctx}}}{A \colon \texttt{Type}, f \colon (A \to A) \to A, x \colon A \vdash x \colon A}}{A \colon \texttt{Type}, f \colon (A \to A) \to A \vdash \lambda x.x \colon A \to A}$$

Finally, we can derive our judgment:

$$\dfrac{\dfrac{A \colon \texttt{Type}, f \colon (A \to A) \to A \vdash f \colon (A \to A) \to A \qquad A \colon \texttt{Type}, f \colon (A \to A) \to A \vdash \lambda x.x \colon A \to A}{A \colon \texttt{Type}, f \colon (A \to A) \to A \vdash f(\lambda x.x) \colon A}}{A \colon \texttt{Type} \vdash \lambda f. f(\lambda x.x) \colon ((A \to A) \to A) \to A}$$

That's it! As a program that computes something, this solution takes a functional and evaluates it on the identity function on A. $\square$

Question 2

I'll put less detail into this one for easy reading.
First of all, let Γ represent the following context:

$$A, B, C\colon \texttt{Type}, f\colon A \to B, g\colon B \to C$$

Then we have the following proof tree:

$$\dfrac{\dfrac{\dfrac{\Gamma \ \texttt{ctx}}{\Gamma, x\colon A \vdash (g \circ f)(x) \equiv (g \circ f)(x)}}{\Gamma, x\colon A \vdash (g \circ f)(x) \equiv \lambda y.g(f(y))(x)}}{\Gamma, x\colon A \vdash (g \circ f)(x) \equiv g(f(x))}$$

To get to the last line, we used the computation rule for function types.
Now let Ψ represent the following context:

$$A, B, C, D\colon \texttt{Type}, f\colon A \to B, g\colon B \to C, h\colon C \to D$$

Using the result of our previous proof tree, we get the following proof tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Psi \ \texttt{ctx}}{\Psi, x\colon A \vdash h(g(f(x))) \equiv h(g(f(x)))\colon D}}{\Psi, x\colon A \vdash (h \circ g)(f(x)) \equiv h((g \circ f)(x))\colon D}}{\Psi, x\colon A \vdash ((h \circ g) \circ f)(x) \equiv (h \circ (g \circ f))(x)\colon D}}{\Psi \vdash \lambda x.((h \circ g) \circ f)(x)) \equiv \lambda x.(h \circ (g \circ f))(x)\colon A \to D}}{\Psi \vdash (h \circ g) \circ f \equiv h \circ (g \circ f)\colon A \to D}$$

The last line follows from the uniqueness principle for function types. □

Question 3

Let $*\colon \mathbf{1}$ be the distinguished element. Then we have the following functions:

$$f :\equiv \lambda x.(*, *)\colon \mathbf{1} \to \mathbf{1} \times \mathbf{1}$$
$$g :\equiv \lambda x.*\colon \mathbf{1} \times \mathbf{1} \to \mathbf{1}$$

The composition $g \circ f$ gives us the following:

$$g \circ f \equiv \lambda x.g(f(x)) \equiv \lambda x.(\lambda y.*)(f(x)) \equiv \lambda x.*\colon \mathbf{1} \to \mathbf{1}$$

Similarly, $f \circ g$ gives us the following:

$$f \circ g \equiv \lambda x.f(g(x)) \equiv \lambda x.(\lambda y.(*, *))(g(x)) \equiv \lambda x.(*, *)\colon \mathbf{1} \times \mathbf{1} \to \mathbf{1} \times \mathbf{1}$$

That's pretty simplified, right? □

Question 4

Let Γ represent the context $P, Q\colon \texttt{Type}$. Then proving De Morgan's law is equivalent to finding an expression E fitting into the following typing judgment:

$$\Gamma \vdash E\colon ((P \to \mathbf{0}) + (Q \to \mathbf{0})) \to (P \times Q) \to \mathbf{0}$$

The following expression works:

$$\lambda f.\lambda a.\texttt{ind}_{P \times Q}(w.\mathbf{0}, x.y.\texttt{ind}_{(P \to \mathbf{0}) + (Q \to \mathbf{0})}(z.\mathbf{0}, g.g(x), h.h(y), f), a)$$

The "classically equivalent statement" corresponds to an expression E fitting into the following typing judgment:

$$\Gamma \vdash E \colon ((P \times Q) \to 0) \to P \to Q \to 0$$

The following (much simpler) expression works for this:

$$\lambda f.\lambda p.\lambda q.f((p, q))$$

That's all! $\square$

Question 1

The following judgement type checks:

$$A : \texttt{Type} \vdash \lambda f.f(\lambda x.x) : ((A \to A) \to A) \to A$$

The proof tree is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{.\ \texttt{ctx}}{(A : \texttt{Type})\ \ \texttt{ctx}}}{A : \texttt{Type} \vdash A : \texttt{Type}}}{A : \texttt{Type} \vdash A \to A : \texttt{Type}} \quad \dfrac{\dfrac{.\ \texttt{ctx}}{(A : \texttt{Type})\ \ \texttt{ctx}}}{A : \texttt{Type} \vdash A : \texttt{Type}}}{A : \texttt{Type} \vdash (A \to A) \to A : \texttt{Type}}}{(A : \texttt{Type}, f : (A \to A) \to A)\ \ \texttt{ctx}}}{A : \texttt{Type}, f : (A \to A) \to A \vdash f : (A \to A) \to A} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{.\ \texttt{ctx}}{(A : \texttt{Type})\ \ \texttt{ctx}}}{A : \texttt{Type} \vdash A : \texttt{Type}}}{A : \texttt{Type} \vdash A \to A : \texttt{Type}} \quad \dfrac{\dfrac{.\ \texttt{ctx}}{(A : \texttt{Type})\ \ \texttt{ctx}}}{A : \texttt{Type} \vdash A : \texttt{Type}}}{A : \texttt{Type} \vdash (A \to A) \to A : \texttt{Type}}}{(A : \texttt{Type}, f : (A \to A) \to A)\ \ \texttt{ctx}}}{A : \texttt{Type}, f : (A \to A) \to A \vdash A : \texttt{Type}}}{(A : \texttt{Type}, f : (A \to A) \to A, x : A)\ \ \texttt{ctx}}}{A : \texttt{Type}, f : (A \to A) \to A, x : A \vdash x : A}}{A : \texttt{Type}, f : (A \to A) \to A \vdash \lambda x.x : A \to A}}{A : \texttt{Type}, f : (A \to A) \to A \vdash f(\lambda x.x) : A}}{A : \texttt{Type} \vdash \lambda f.f(\lambda x.x) : ((A \to A) \to A) \to A}$$

If you think of the expression

$$E \equiv \lambda f.f(\lambda x.x)$$

as a functional program that computes something, it take as input a function and applies it to an identity function for type A. This results in a return value of type A.

`Coq` code:

```
Theorem problem1 (A:Type) : ((A → A) → A) → A.
Proof.
    intro f.
    apply f.
    intro x.
    exact x.
Qed.
```

Question 2

Given the context

$$\Gamma = (A, B, C, D : \texttt{Type}, f : A \to B, g : B \to C, h : C \to D)$$

Show that function composition is associative up to judgemental equality:

$$(h \circ g) \circ f \equiv h \circ (g \circ f) : A \to C$$

by constructing a proof tree of the following:

$$\Gamma \vdash \lambda y.\lambda x.h(g(x))(f(y)) \equiv \lambda y.h(\lambda x.g(f(x))(y)) : A \to D$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\text{see (a) below}\\ \vdots\end{matrix}}{\Gamma, y : A, x : B \vdash h(g(x)) : D} \quad \dfrac{\begin{matrix}\text{see (b) below}\\ \vdots\end{matrix}}{\Gamma, y : A \vdash f(y) : B}}{\Gamma, y : A \vdash \lambda x.h(g(x))(f(y)) \equiv h(g(f(y))) : D}}{\Gamma \vdash \lambda y.\lambda x.h(g(x))(f(y)) \equiv \lambda y.h(g(f(y))) : A \to D} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \quad \mathtt{ctx}}}{\Gamma \vdash A : \mathtt{Type}}}{(\Gamma, y : A) \ \ \mathtt{ctx}}}{\Gamma, y : A \vdash h \equiv h : C \to D} \quad \dfrac{\dfrac{\begin{matrix}\text{see (c) below}\\ \vdots\end{matrix}}{\Gamma, y : A, x : A \vdash g(f(x)) : C} \quad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \quad \mathtt{ctx}}}{\Gamma \vdash A : \mathtt{Type}}}{(\Gamma, y : A) \ \ \mathtt{ctx}}}{\Gamma, y : A \vdash y : A}}{\Gamma, y : A \vdash g(f(y)) \equiv \lambda x.g(f(x))(y) : C}}{\Gamma, y : A \vdash h(g(f(y))) \equiv h(\lambda x.g(f(x))(y)) : D}}{\Gamma \vdash \lambda y.h(g(f(y))) \equiv \lambda y.h(\lambda x.g(f(x))(y)) : A \to D}}{\Gamma \vdash \lambda y.\lambda x.h(g(x))(f(y)) \equiv \lambda y.h(\lambda x.g(f(x))(y)) : A \to D}$$

subtree (a):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \quad \mathtt{ctx}}}{\Gamma \vdash A : \mathtt{Type}}}{(\Gamma, y : A) \ \ \mathtt{ctx}}}{\Gamma, y : A \vdash B : \mathtt{Type}}}{(\Gamma, y : A, x : B) \ \ \mathtt{ctx}}}{\Gamma, y : A, x : B \vdash h : C \to D} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \quad \mathtt{ctx}}}{\Gamma \vdash A : \mathtt{Type}}}{(\Gamma, y : A) \ \ \mathtt{ctx}}}{\Gamma, y : A \vdash B : \mathtt{Type}}}{(\Gamma, y : A, x : B) \ \ \mathtt{ctx}}}{\Gamma, y : A, x : B \vdash g : B \to C} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \quad \mathtt{ctx}}}{\Gamma \vdash A : \mathtt{Type}}}{(\Gamma, y : A) \ \ \mathtt{ctx}}}{\Gamma, y : A \vdash B : \mathtt{Type}}}{(\Gamma, y : A, x : B) \ \ \mathtt{ctx}}}{\Gamma, y : A, x : B \vdash x : B}}{\Gamma, y : A, x : B \vdash g(x) : C}}{\Gamma, y : A, x : B \vdash h(g(x)) : D}$$

subtree (b):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \quad \mathtt{ctx}}}{\Gamma \vdash A : \mathtt{Type}}}{(\Gamma, y : A) \ \ \mathtt{ctx}}}{\Gamma, y : A \vdash f : A \to B} \quad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \quad \mathtt{ctx}}}{\Gamma \vdash A : \mathtt{Type}}}{(\Gamma, y : A) \ \ \mathtt{ctx}}}{\Gamma, y : A \vdash y : A}}{\Gamma, y : A \vdash f(y) : B}$$

subtree (c):

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\UnaryInfC{$\Gamma \quad \mathtt{ctx}$}
\UnaryInfC{$\Gamma \vdash A : \mathtt{Type}$}
\UnaryInfC{$(\Gamma, y : A) \ \ \mathtt{ctx}$}
\UnaryInfC{$\Gamma, y : A \vdash A : \mathtt{Type}$}
\UnaryInfC{$(\Gamma, y : A, x : A) \ \ \mathtt{ctx}$}
\UnaryInfC{$\Gamma, y : A, x : A \vdash g : B \to C$}
\AxiomC{$\vdots$}
\UnaryInfC{$\Gamma \quad \mathtt{ctx}$}
\UnaryInfC{$\Gamma \vdash A : \mathtt{Type}$}
\UnaryInfC{$(\Gamma, y : A) \ \ \mathtt{ctx}$}
\UnaryInfC{$\Gamma, y : A \vdash A : \mathtt{Type}$}
\UnaryInfC{$(\Gamma, y : A, x : A) \ \ \mathtt{ctx}$}
\UnaryInfC{$\Gamma, y : A, x : A \vdash f : A \to B$}
\AxiomC{$\vdots$}
\UnaryInfC{$\Gamma \quad \mathtt{ctx}$}
\UnaryInfC{$\Gamma \vdash A : \mathtt{Type}$}
\UnaryInfC{$(\Gamma, y : A) \ \ \mathtt{ctx}$}
\UnaryInfC{$\Gamma, y : A \vdash A : \mathtt{Type}$}
\UnaryInfC{$(\Gamma, y : A, x : A) \ \ \mathtt{ctx}$}
\UnaryInfC{$\Gamma, y : A, x : A \vdash x : A$}
\BinaryInfC{$\Gamma, y : A, x : A \vdash f(x) : B$}
\BinaryInfC{$\Gamma, y : A, x : A \vdash g(f(x)) : C$}
\end{prooftree}
$$

Question 3

Consider the functions

$$f \equiv \lambda(x : 1).(x, x) : \mathbf{1} \to \mathbf{1} \times \mathbf{1}$$

$$g \equiv \lambda p.\mathtt{ind}_{\mathbf{1}\times\mathbf{1}}(z.\mathbf{1}, a.b.a, p) : \mathbf{1} \times \mathbf{1} \to \mathbf{1}$$

The composition of these functions are:

$$g \circ f \equiv \lambda u.(\lambda p.\mathtt{ind}_{\mathbf{1}\times\mathbf{1}}(z.\mathbf{1}, a.b.a, p))((\lambda x.(x, x))(u)) : 1 \to 1$$

$$f \circ g \equiv \lambda q.(\lambda x.(x, x))(\lambda p.\mathtt{ind}_{\mathbf{1}\times\mathbf{1}}(z.\mathbf{1}, a.b.a, p)(q)) : 1 \times 1 \to 1 \times 1$$

By applying the computation rule for function types twice, we get the following simplifications:

$$g \circ f \equiv \lambda u.\mathtt{ind}_{\mathbf{1}\times\mathbf{1}}(z.\mathbf{1}, a.b.a, (u, u)),$$

$$f \circ g \equiv \lambda q.(\mathtt{ind}_{\mathbf{1}\times\mathbf{1}}(z.\mathbf{1}, a.b.a, q), \mathtt{ind}_{\mathbf{1}\times\mathbf{1}}(z.\mathbf{1}, a.b.a, q)).$$

We can simplifiy $g \circ f$ even further by applying the computation rule for products:

$$g \circ f \equiv \lambda q.q$$

Note that this is just the identity function.

Question 4

The statement

"if (not P) or (not Q), then not (P and Q)"

corresponds to the following type:

$$((P \to \mathbf{0}) + (Q \to \mathbf{0})) \to (P \times Q) \to \mathbf{0}$$

A proof of the statement is a program of the above type:

$$\lambda f.\lambda g.\mathtt{ind}_{(P\to\mathbf{0})+(Q\to\mathbf{0})}(z.\mathbf{0}, x.\mathtt{ind}_{P\times Q}(p.\mathbf{0}, a.b.x(a), g), y.\mathtt{ind}_{P\times Q}(p.\mathbf{0}, a.b.y(b), g), f)$$

The proof tree below illustrates a proof of this statement using the following abbreviations:

$$\begin{aligned}
E_1 &= \mathtt{ind}_{P\times Q}(p.\mathbf{0}, a.b.x(a), g)\\
E_2 &= \mathtt{ind}_{P\times Q}(p.\mathbf{0}, a.b.y(b), g)\\
\Gamma &= (P : \mathtt{Type}, Q : \mathtt{Type}, f : (P\to 0) + (Q\to 0), g : P\times Q)\\
\Gamma_2 &= (P : \mathtt{Type}, Q : \mathtt{Type}, f : (P\to 0) + (Q\to 0), g : P\times Q, a : P, b : Q)
\end{aligned}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\overline{.\ \mathtt{ctx}}\\ \vdots\end{array}}{\Gamma\ \ \mathtt{ctx}}}{\Gamma\vdash\mathbf{0}:\mathtt{Type}} \quad \dfrac{\begin{array}{c}\text{see subtree (a)}\\ \vdots\end{array}}{\Gamma, x:(P\to\mathbf{0})\vdash E_1:\mathbf{0}} \quad \dfrac{\begin{array}{c}\text{see subtree (b)}\\ \vdots\end{array}}{\Gamma, y:(Q\to\mathbf{0})\vdash E_2:\mathbf{0}} \quad \dfrac{\dfrac{\begin{array}{c}\overline{.\ \mathtt{ctx}}\\ \vdots\end{array}}{\Gamma\ \ \mathtt{ctx}}}{\Gamma\vdash f:((P\to 0)+(Q\to 0))}}{P:\mathtt{Type}, Q:\mathtt{Type}, f:(P\to 0)+(Q\to 0), g:P\times Q\vdash \mathtt{ind}_{(P\to\mathbf{0})+(Q\to\mathbf{0})}(z.\mathbf{0}, x.E_1, y.E_2, f):\mathbf{0}}}{P:\mathtt{Type}, Q:\mathtt{Type}, f:(P\to 0)+(Q\to 0)\vdash \lambda g.\mathtt{ind}_{(P\to\mathbf{0})+(Q\to\mathbf{0})}(z.\mathbf{0}, x.E_1, y.E_2, f):(P\times Q)\to\mathbf{0}}}{P:\mathtt{Type}, Q:\mathtt{Type}\vdash \lambda f.\lambda g.\mathtt{ind}_{(P\to\mathbf{0})+(Q\to\mathbf{0})}(z.\mathbf{0}, x.E_1, y.E_2, f):((P\to\mathbf{0})+(Q\to\mathbf{0}))\to(P\times Q)\to\mathbf{0}}$$

subtree (a):

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\overline{.\ \mathtt{ctx}}\\ \vdots\end{array}}{\Gamma\ \ \mathtt{ctx}}}{\Gamma\vdash\mathbf{0}:\mathtt{Type}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\overline{.\ \mathtt{ctx}}\\ \vdots\end{array}}{\Gamma_2\ \ \mathtt{ctx}}}{\Gamma_2\vdash P:\mathtt{Type}} \quad \dfrac{\dfrac{\begin{array}{c}\overline{.\ \mathtt{ctx}}\\ \vdots\end{array}}{\Gamma_2\ \ \mathtt{ctx}}}{\Gamma_2\vdash\mathbf{0}:\mathtt{Type}}}{\Gamma_2\vdash P\to\mathbf{0}:\mathtt{Type}}}{(\Gamma_2, x:P\to\mathbf{0})\ \ \mathtt{ctx}}}{\Gamma_2, x:P\to\mathbf{0}\vdash x:P\to\mathbf{0}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\overline{.\ \mathtt{ctx}}\\ \vdots\end{array}}{\Gamma_2\ \ \mathtt{ctx}}}{\Gamma_2\vdash P:\mathtt{Type}} \quad \dfrac{\dfrac{\begin{array}{c}\overline{.\ \mathtt{ctx}}\\ \vdots\end{array}}{\Gamma_2\ \ \mathtt{ctx}}}{\Gamma_2\vdash\mathbf{0}:\mathtt{Type}}}{\Gamma_2\vdash P\to\mathbf{0}:\mathtt{Type}}}{(\Gamma_2, x:P\to\mathbf{0})\ \ \mathtt{ctx}}}{\Gamma_2, x:P\to\mathbf{0}\vdash a:P}}{\Gamma, x:(P\to\mathbf{0}), a:P, b:Q\vdash x(a):\mathbf{0}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\overline{.\ \mathtt{ctx}}\\ \vdots\end{array}}{\Gamma\ \ \mathtt{ctx}}}{\Gamma\vdash P:\mathtt{Type}} \quad \dfrac{\dfrac{\begin{array}{c}\overline{.\ \mathtt{ctx}}\\ \vdots\end{array}}{\Gamma\ \ \mathtt{ctx}}}{\Gamma\vdash Q:\mathtt{Type}}}{\Gamma, x:(P\to\mathbf{0})\vdash P\times Q:\mathtt{Type}}}{\Gamma, x:(P\to\mathbf{0})\vdash g:P\times Q}}{\Gamma, x:(P\to\mathbf{0})\vdash \mathtt{ind}_{P\times Q}(p.\mathbf{0}, a.b.x(a), g):\mathbf{0}}$$

subtree (b)

$$\dfrac{\dfrac{\dfrac{\dfrac{.\ \mathtt{ctx}}{\vdots}}{\Gamma\ \ \mathtt{ctx}}}{\Gamma \vdash \mathbf{0} : \mathtt{Type}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{.\ \mathtt{ctx}}{\vdots}}{\Gamma_2\ \ \mathtt{ctx}}}{\Gamma_2 \vdash Q : \mathtt{Type}} \quad \dfrac{\dfrac{\dfrac{.\ \mathtt{ctx}}{\vdots}}{\Gamma_2\ \ \mathtt{ctx}}}{\Gamma_2 \vdash \mathbf{0} : \mathtt{Type}}}{\Gamma_2 \vdash Q \to \mathbf{0} : \mathtt{Type}}}{(\Gamma_2, y : Q \to \mathbf{0})\ \ \mathtt{ctx}}}{\Gamma_2, y : Q \to \mathbf{0} \vdash y : Q \to \mathbf{0}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{.\ \mathtt{ctx}}{\vdots}}{\Gamma_2\ \ \mathtt{ctx}}}{\Gamma_2 \vdash Q : \mathtt{Type}} \quad \dfrac{\dfrac{\dfrac{.\ \mathtt{ctx}}{\vdots}}{\Gamma_2\ \ \mathtt{ctx}}}{\Gamma_2 \vdash \mathbf{0} : \mathtt{Type}}}{\Gamma_2 \vdash Q \to \mathbf{0} : \mathtt{Type}}}{(\Gamma_2, y : Q \to \mathbf{0})\ \ \mathtt{ctx}}}{\Gamma_2, y : Q \to \mathbf{0} \vdash b : Q}}{\Gamma, y : (Q \to \mathbf{0}), a : P, b : Q \vdash y(b) : \mathbf{0}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{.\ \mathtt{ctx}}{\vdots}}{\Gamma\ \ \mathtt{ctx}}}{\Gamma \vdash P : \mathtt{Type}} \quad \dfrac{\dfrac{\dfrac{.\ \mathtt{ctx}}{\vdots}}{\Gamma\ \ \mathtt{ctx}}}{\Gamma \vdash Q : \mathtt{Type}}}{\Gamma, y : (Q \to \mathbf{0}) \vdash P \times Q : \mathtt{Type}}}{\Gamma, y : (Q \to \mathbf{0}) \vdash g : P \times Q}}{\Gamma, y : (Q \to \mathbf{0}) \vdash \mathtt{ind}_{P \times Q}(p.\mathbf{0}, a.b.y(b), g) : \mathbf{0}}$$

The statement

"if not (P and Q), then P implies (not Q)"

corresponds to the following type:

$$((P \times Q) \to \mathbf{0}) \to (P \to (Q \to \mathbf{0}))$$

A proof of the statement is a program of the above type:

$$\lambda f.\lambda x.\lambda y.f(x, y)$$

The following proof tree illustrates a proof of this statement using the following abbreviation:

$$\Gamma = (P : \mathtt{Type}, Q : \mathtt{Type}, f : (P \times Q) \to \mathbf{0}, x : P, y : Q)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{.\ \mathtt{ctx}}{\vdots}}{\Gamma\ \ \mathtt{ctx}}}{\Gamma \vdash f : (P \times Q) \to \mathbf{0}} \quad \dfrac{\dfrac{\dfrac{\dfrac{.\ \mathtt{ctx}}{\vdots}}{\Gamma\ \ \mathtt{ctx}}}{\Gamma \vdash x : P} \quad \dfrac{\dfrac{\dfrac{.\ \mathtt{ctx}}{\vdots}}{\Gamma\ \ \mathtt{ctx}}}{\Gamma \vdash y : Q}}{\Gamma \vdash (x, y) : P \times Q}}{P : \mathtt{Type}, Q : \mathtt{Type}, f : (P \times Q) \to \mathbf{0}, x : P, y : Q \vdash f(x, y) : \mathbf{0}}}{P : \mathtt{Type}, Q : \mathtt{Type}, f : (P \times Q) \to \mathbf{0}, x : P \vdash \lambda y.f(x, y) : Q \to \mathbf{0}}}{P : \mathtt{Type}, Q : \mathtt{Type}, f : (P \times Q) \to \mathbf{0} \vdash \lambda x.\lambda y.f(x, y) : P \to (Q \to \mathbf{0})}}{P : \mathtt{Type}, Q : \mathtt{Type} \vdash \lambda f.\lambda x.\lambda y.f(x, y) : ((P \times Q) \to \mathbf{0}) \to (P \to (Q \to \mathbf{0}))}$$

Proof of $\Gamma\ \ \mathtt{ctx}$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{.\ \mathtt{ctx}}{(P:\mathsf{Type})\ \mathtt{ctx}}}{(P:\mathsf{Type},Q:\mathsf{Type})\ \mathtt{ctx}}}{P:\mathsf{Type},Q:\mathsf{Type}\vdash\mathbf{0}:\mathsf{Type}} \quad \dfrac{\dfrac{\dfrac{\dfrac{.\ \mathtt{ctx}}{(P:\mathsf{Type})\ \mathtt{ctx}}}{(P:\mathsf{Type},Q:\mathsf{Type})\ \mathtt{ctx}}}{P:\mathsf{Type},Q:\mathsf{Type}\vdash P:\mathsf{Type}} \quad \dfrac{\dfrac{\dfrac{.\ \mathtt{ctx}}{(P:\mathsf{Type})\ \mathtt{ctx}}}{(P:\mathsf{Type},Q:\mathsf{Type})\ \mathtt{ctx}}}{P:\mathsf{Type},Q:\mathsf{Type}\vdash Q:\mathsf{Type}}}{P:\mathsf{Type},Q:\mathsf{Type}\vdash P\times Q:\mathsf{Type}}}{P:\mathsf{Type},Q:\mathsf{Type}\vdash(P\times Q)\to\mathbf{0}):\mathsf{Type}}}{(P:\mathsf{Type},Q:\mathsf{Type},f:(P\times Q)\to\mathbf{0})\ \mathtt{ctx}}}{P:\mathsf{Type},Q:\mathsf{Type},f:(P\times Q)\to\mathbf{0}\vdash P:\mathsf{Type}\ \ \mathtt{ctx}}}{(P:\mathsf{Type},Q:\mathsf{Type},f:(P\times Q)\to\mathbf{0},x:P)\ \mathtt{ctx}}}{P:\mathsf{Type},Q:\mathsf{Type},f:(P\times Q)\to\mathbf{0},x:P:Q:\mathsf{Type}}}{(P:\mathsf{Type},Q:\mathsf{Type},f:(P\times Q)\to\mathbf{0},x:P,y:Q)\ \mathtt{ctx}}
$$

Coq code:

```
Theorem DeMorganA (P Q:Type) : ((P → False) + (Q → False)) → (P * Q) → False.
Proof.
    intro f.
    intro p.
    destruct f.
    destruct p.
    exact (f p).
    destruct p.
    exact (f q).
Qed.

Theorem DeMorganB (P Q:Type) : ((P * Q) → False) → (P → (Q → False)).
Proof.
    intro f.
    intro x.
    intro y.
    exact (f (x,y)).
Qed.
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