Curious properties of iterated measurements

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Motivation

How do we know that the world is quantum-mechanical?

E.g.:

- Rule out classical theories: we observe violations of the Bell inequality.
- Validate quantum theory: the Tsirelson bound holds in experiment!

But why is the world quantum-mechanical? **Idea:**

(a) Determine which kinds of correlations are quantum-mechanical (as exemplified e.g. by the Tsirelson bound),

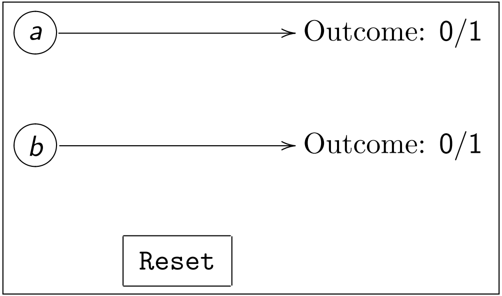
(b) and then try to find a physically intuitive reason why correlations of this kind are observed.

This would explain why our world is quantum-mechanical.

Here, we will discuss a particular case of step (a).

Setting: two iterated dichotomic measurements

Given a "black box" experiment with two possible measurements, each having two outcomes,



we can experimentally determine values for outcome probabilities of iterated measurements:

- iterated measurement:
  e.g.: reset to initial state, measure a, then b, then a, then a.
- outcome probability: e.g. $P_{a,b,a,a}(0,1,1,1)$.

The representation problem of quantum measurements

- **Question:** When is it possible that these probabilities

$$P_{a,b,a,a}(0,1,1,1),\ P_{b,a,b}(0,0,1),\ \ldots$$

  come from a quantum-mechanical description of the system?
- **Assumption:** All measurements are repeatable: directly consecutive identical measurements always give the same result.
  $\Rightarrow$ There is no dynamics between measurements.
- So we look for

$$|\psi\rangle \in \mathcal{H}, \quad \text{projection operators } \ a, b \in \mathcal{B}(\mathcal{H})$$

  which reproduce the probabilities via the usual rules of quantum mechanics without dynamics.
- This is the simplest non-trivial case of the representation problem for iterated measurements.

A curious necessary condition

Theorem

In every quantum-mechanical description of the system,

$$P_{a,b,a}(0,0,1) = P_{a,b,a}(0,1,1) \tag{1}$$

Proof.

The relevant ingredient is $a(1-a) = 0$. We have,

$$\begin{aligned}
P_{a,b,a}(0,0,1) &= ||a(1-b)(1-a)|\psi\rangle||^2 \\
&= \langle\psi|(1-a)(1-b)a(1-b)(1-a)|\psi\rangle \\
&= -\langle\psi|(1-a)ba(1-b)(1-a)|\psi\rangle \\
&= \langle\psi|(1-a)bab(1-a)|\psi\rangle = P_{a,b,a}(0,1,1).
\end{aligned}$$

□

Digression on $P_{a,b,a}(0,0,1) = P_{a,b,a}(0,1,1)$

More generally, consider an experiment where we do

(1) preparation of a state $|\psi_i\rangle$,

(2) measurement of a projection operator p,

(3) post-selection for a state $|\psi_f\rangle$,

such that $\langle\psi_f|\psi_i\rangle = 0$.

Theorem

Then, if $\langle\psi_f|p|\psi_i\rangle \neq 0$,

$$P_p(0) = P_p(1) = \frac{1}{2}.$$

Digression: a variant of Aharonov's 3-box experiment

Let $\zeta = e^{\frac{2\pi i}{3}}$, so that $1 + \zeta + \zeta^2 = 0$.

Example

Particle that can be in one of three boxes, so that $\mathcal{H} = \mathbb{C}^3$.

$$\text{pre-selection: } |\psi_i\rangle = \frac{|1\rangle + |2\rangle + |3\rangle}{\sqrt{3}}$$

box $\|1\rangle$	box $\|2\rangle$	box $\|3\rangle$

$$\text{post-selection: } |\psi_f\rangle = \frac{|1\rangle + \zeta|2\rangle + \zeta^2|3\rangle}{\sqrt{3}}$$

Upon opening each one of the boxes, the probability of finding it there is $\frac{1}{2}$!

Higher order conditions

Given a binary sequence $(r_i)_i \in \{0,1\}^n$, let $s(r_i)$ be the number of switches in the sequence.

Theorem

In every quantum-mechanical description, the probabilities

$$P_{a,b,a,\ldots}(r_1, \ldots, r_n) \tag{2}$$

depend only on r_1 and $s(r_i)$.

This essentially follows from

$$P_{a,b,a}(0,0,1) = P_{a,b,a}(0,1,1)$$

together with the analogous equations with $a \leftrightarrow b$ interchanged or $0 \leftrightarrow 1$ interchanged. E.g. for an $n = 6$ sequence with 3 switches:

$$001101 \to 011101 \to 011001 \to 010001 \to 010011 \to 010111$$

More curious identities

Theorem

In every quantum-mechanical description of the system,

$$P_{a,b}(0,0) + P_{a,b}(1,1) = P_{b,a}(0,0) + P_{b,a}(1,1)$$

Proof.

The expression

$$\begin{aligned} & P_{a,b}(0,0) + P_{a,b}(1,1) \\ &= ||(1-b)(1-a)|\psi\rangle||^2 + ||ba|\psi\rangle||^2 \\ &= \langle\psi|(1-b)(1-a)(1-b) + bab\,|\psi\rangle \\ &= \langle\psi|1 + \{a,b\} - a - b\,|\psi\rangle \end{aligned}$$

is invariant under $a \leftrightarrow b$. □

More higher-order conditions

This generalizes to:

Theorem

In every quantum-mechanical description of the system,

$$\begin{aligned} & P_{a,b,a,\ldots}(r_1,\ldots,r_n) + P_{a,b,a,\ldots}(\overline{r_1},\ldots,\overline{r_n}) \\ = \; & P_{b,a,b,\ldots}(r_1,\ldots,r_n) + P_{b,a,b,\ldots}(\overline{r_1},\ldots,\overline{r_n}) \end{aligned}$$

- This ends the list of equational conditions.
- There is also a hierarchy of complicated inequality conditions.
- For the full long-winded classification theorem, see `http://arxiv.org/abs/0908.2559`.

Systematic approach: basic ideas I

- Let
$$\mathcal{A} = C^*\left(a, b \,\middle|\, a^* = a = a^2,\ b^* = b = b^2\right)$$
be the universal C^*-algebra freely generated by two projections.
- It is well-known that
$$\mathcal{A} \cong \left\{ f : [0,1] \xrightarrow{\text{cont.}} M_2(\mathbb{C}) \,\middle|\, f(0) \text{ and } f(1) \text{ are diagonal} \right\}$$
where the projections are given by the functions
$$a(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b(x) = \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix}.$$

Systematic approach: basic ideas II

- By the universal property of the C^*-algebra $\mathcal{A}$, any quantum-mechanical description of the "black box" is equivalent to the specification of a C^*-algebraic state

$$\rho : \mathcal{A} \to \mathbb{C}$$

  In turn, the GNS construction turns any C^*-algebraic state on $\mathcal{A}$ into a quantum-mechanical model.
- This reduces the problem to a certain noncommutative moment problem on $\mathcal{A}$.
- This moment problem can be solved by using the relations (with $\overline{a} = 1 - a$, $\overline{b} = 1 - b$)

$$\begin{aligned} aba &= xa, & bab &= xb \\ a\overline{b}a &= (1-x)a, & b\overline{a}b &= (1-x)b \\ \overline{a}b\overline{a} &= (1-x)\overline{a}, & \overline{b}a\overline{b} &= (1-x)\overline{b} \\ \overline{a}\overline{b}\overline{a} &= x\overline{a}, & \overline{b}\overline{a}\overline{b} &= x\overline{b} \end{aligned}$$

General probabilistic theories I

Are the conditions found really specific for QM or do they hold more generally?

Definition

A general probabilistic model is given by a vector space V, a cone of unnormalized states $\Omega \subseteq V$ and a linear functional

$$\mathrm{tr} : V \to \mathbb{R}$$

such that $\mathrm{tr}(\omega) > 0$ for $\omega \in \Omega$, $\omega \neq 0$.
An operation is a linear map $a : V \to V$ such that $a(\Omega) \subseteq \Omega$ and $\mathrm{tr}(a(\omega)) \leq \mathrm{tr}(\omega)$ for $\omega \in \Omega$.

Inituitively, $\mathrm{tr}(a(\omega))$ is the probability that the operation a happens in the state ω with normalization $\mathrm{tr}(\omega) = 1$.

General probabilistic theories II

Definition

A dichotomic measurement consists of two operations $a, \overline{a} : V \longrightarrow V$ *which are idempotent,*

$$a^2 = a, \qquad \overline{a}^2 = \overline{a}$$

such that $a + \overline{a}$ *is* tr*-preserving.*

Then the probability of measuring 1 or 0 on ω with $\mathrm{tr}(\omega) = 1$ are given by, respectively,

$$\mathrm{tr}(a(\omega)), \qquad \mathrm{tr}(\overline{a}(\omega)).$$

Preservation of tr guarantess that these sum up to 1.

General probabilistic theories III

Given dichotomic measurements a vs. $\bar{a}$ and b vs. $\bar{b}$ and an initial state ω with $\mathrm{tr}(\omega) = 1$, the outcome probabilities are given by expressions like

$$P_{a,b,a}(0,0,1) = \mathrm{tr}(\bar{a}ba(\omega))$$

Theorem

Given any choice of probability assignments for the $P_{a,b,a,\ldots}$ and the $P_{b,a,b,\ldots}$, there exists a general probabilistic model reproducing these probabilities on some ω.

Idea of proof: same as in the quantum case—consider a certain universal algebra and states on it.

Finite-dimensional truncations I

In actual experiments, the following issues arise:

- perfect von Neumann measurements do not exist,
- errors due to finite statistics,
- only a finite subset of the probabilities is known.

Here, concentrate on the latter problem. So what is the quantum region in a finite-dimensional *truncation* of probability space? The full classification theorem is not very helpful for answering this question.

$\Rightarrow$ Need another method for determining the quantum region in a finite truncation:

- Any state on $\mathcal{A}$ is in the closure of the convex hull of pure states.
- Going from state space to probability space is a linear map.
- $\Rightarrow$ The quantum region in the truncation is the convex hull of outcome probabilities on pure states.

Finite-dimensional truncations II

The pure states on $\mathcal{A}$ are

$$\mathcal{A} \to \mathbb{C}, \quad f \mapsto \langle\psi|f(t_0)|\psi\rangle \quad \text{with} \quad |\psi\rangle = \begin{pmatrix} \cos\theta \\ e^{i\lambda}\sin\theta \end{pmatrix}$$

with $t_0 \in [0,1]$ and $\theta, \lambda \in [0, 2\pi]$.

Conclusion: The quantum region in some finite-dimensional truncation is the convex hull of an explicitly given algebraic variety in Euclidean space.

Towards experimental tests of the quantum constraints

Obvious requirements on candidate physical systems:

- Two dichotomic von Neumann measurements are available.
- No additional dynamics.
- Measurement of these observables does not destroy the system.
- A sufficiently big system is needed to allow for degeneracy,
- so that a two-state system is not sufficient.

Summary

- Quantum-mechanical probabilities for outcomes of two dichotomic measurements show unexpected relations, for example

$$P_{a,b,a}(0,0,1) = P_{a,b,a}(0,1,1)$$

and

$$P_{a,b}(0,0) + P_{a,b}(1,1) = P_{b,a}(0,0) + P_{b,a}(1,1).$$

- These relations do not generally hold in general probabilistic theories.
- This might facilitate further experimental discrimination between quantum theory and other general probabilistic models.
- An intuitive physical explanation for the presence of these relations seems hard to find.