Abstract Convexity: Results and Speculations

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Overview

Part 1: Introducing convex spaces

Introduction

Historically, the paradigm shifts

concrete submanifolds of $\mathbb{R}^n$ $\rightarrow$ abstract manifolds

concrete sets of symmetries $\rightarrow$ abstract groups

were very important for the development of mathematics.

Question

Does the concept of **convexity** also admit an abstract counterpart?

$\rightarrow$ Positive answer: theory of convex spaces. ("abstract convexity")

Definition (write $\overline{\lambda} = 1 - \lambda$)

A **convex space** is a set C together with a family of binary operations

$$C \times C \longrightarrow C, \quad (x, y) \mapsto \lambda x + \overline{\lambda} y, \quad \lambda \in (0, 1)$$

satisfying the conditions

- idempotence: $\lambda x + \overline{\lambda} x = x$
- parametric commutativity: $\lambda y + \overline{\lambda} x = \overline{\lambda} x + \overline{\overline{\lambda}} y$
- deformed parametric associativity:

$$\lambda (\mu x + \overline{\mu} y) + \overline{\lambda} z = \lambda' x + \overline{\lambda'} (\mu' y + \overline{\mu'} z)$$

  with

$$\lambda' = \lambda\mu, \qquad \mu' = \frac{\lambda(1 - \mu)}{1 - \lambda\mu}$$

Definition

A map $f : C \to D$ is convex if $f(\lambda x + \overline{\lambda} y) = \lambda f(x) + \overline{\lambda} f(y)$.

Features of abstract convexity theory

- Philosophy: study a convex set in terms of its internal structure, forgetting that it lies in a vector space.
- This reveals a rich intrinsic geometry of convex sets.
- It unifies aspects of convex geometry and functional analysis (Banach spaces) with aspects of order theory (semilattices).
- It provides a nice category of convex sets:
  symmetric monoidal closed, complete and cocomplete.

Example

A convex subset of a vector space is a convex space.

A convex space that can be written in this form is called **geometric**.

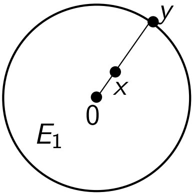
Example

Let $(E, ||\cdot||)$ be a normed vector space. Then the unit ball

$$E_1 \equiv \{x \in E \mid ||x|| \leq 1\}$$

is a convex space. It determines the norm of $x \in E_1$ via

$$||x|| = \inf\{r \in [0,1] \mid \exists y \in E_1,\ x = ry + \overline{r}\,0\}$$



Characterizing geometric convex spaces

Theorem

A convex space C is geometric if and only if it satisfies the cancellation condition

$$\lambda x + \overline{\lambda} y = \lambda x' + \overline{\lambda}\, y \quad \Longrightarrow \quad x = x'$$

for all $x, x', y \in C$.

Sketch of proof.

Construct a canonical linear hull $\mathrm{lin}(C)$ using formal differences of points of C. If the cancellation condition holds, the canonical map $C \to \mathrm{lin}(C)$ is injective. □

The convex space of all polytopes

Example

Given two polytopes $P, Q \subseteq \mathbb{R}^n$, their convex combination can be defined, in analogy with Minkowski sum, as the new polytope

$$\lambda P + \overline{\lambda} Q \equiv \left\{ \lambda x + \overline{\lambda} y,\ x \in P,\ y \in Q \right\} .$$

$\Rightarrow$ The set of polytopes in $\mathbb{R}^n$ is a convex space in its own right. Since the cancellation law holds for polytopes[1], the convex space of polytopes can be embedded into a vector space.

[1] Sketch of proof: suppose that $\lambda P + (1-\lambda)Q = \lambda P' + (1-\lambda)Q$. This implies $n\lambda P + (1-\lambda)Q = n\lambda P' + (1-\lambda)Q$ for any $n \in \mathbb{N}$. Then $P = P'$ follows by choosing n large enough.

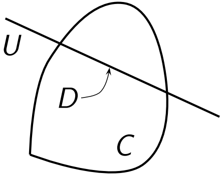
Part 2: A Hahn-Banach theorem and convex homotopy?

Pseudo-extremal subspaces

Definition

A convex subset $D \subseteq C$ is called **pseudo-extremal** if for all $x, y \in C$, $\lambda \in (0,1)$,

$$x \in D,\ \lambda x + \overline{\lambda} y \in D \ \Rightarrow\ y \in D$$



Theorem

When $C \subseteq V$ is geometric, then the pseudo-extremal subspaces are those of the form

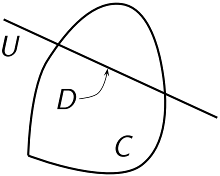
$$D = C \cap U$$

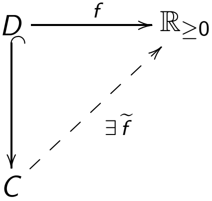
for $U \subseteq V$ an affine subspace.

An abstract Hahn-Banach theorem

Theorem

For geometric C and pseudo-extremal $D \subseteq C$ and a convex functional $f : D \to \mathbb{R}_{\geq 0}$, there is a convex functional $\widetilde{f} : C \to \mathbb{R}_{\geq 0}$ extending f.





Idea of proof.

Take $D = U \cap C$, with $U \subseteq V$ an affine subspace. Use Zorn's lemma to reduce to the case where $U \subseteq V$ has codimension 1. The rest is a geometrical argument also similar to the classical proof. □

Classical Hahn-Banach as a special case

Theorem

$(E, ||\cdot||)$ a normed vector space, $F \subseteq E$ a subspace, and $\varphi : F \to \mathbb{R}$ linear bounded.

$\Rightarrow$ There is a linear $\widetilde{\varphi} : E \to \mathbb{R}$ with

$$\widetilde{\varphi}_{|F} = \varphi, \qquad ||\widetilde{\varphi}|| = ||\varphi||.$$

Sketch of proof.

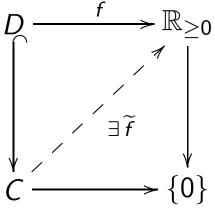
Apply the abstract Hahn-Banach theorem to the unit balls $D = F_1 \subseteq E_1 = C$ and the functional

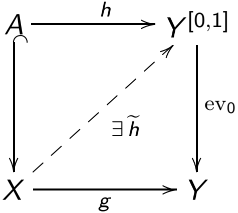
$$f(x) = \varphi(x) + ||\varphi||.$$

Then $\widetilde{\varphi}(x) = \widetilde{f}(x) - ||\varphi||$ does the job. □

The homotopy extension property

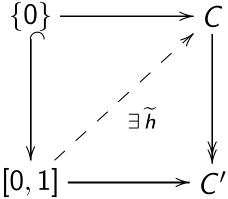
The abstract Hahn-Banach theorem resembles the **homotopy extension property** of topological spaces:

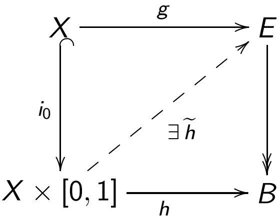




The homotopy lifting property

For any convex surjection $C \twoheadrightarrow C'$, there is also an analog of the **homotopy lifting property**:





Towards a homotopy theory of convex spaces?

The lifting and extension properties are needed for getting nice homotopical invariants of topological spaces.

Question

Is there a homotopy theory of convex spaces?

$\Longrightarrow$ Try to find a nice **model category structure** on the category of convex spaces.

A model category structure consists of classes of morphisms,

- fibrations,
- cofibrations,
- weak equivalences,

satisfying certain conditions, including extension/lifting properties.

By the Hahn-Banach theorem: cofibrations should be given by inclusions of pseudo-extremal subspaces.

Why algebraic invariants of convex spaces?

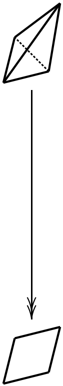
Goal of convex homotopy: find algebraic invariants of convex spaces!

These could help in deciding

- lifting problems for polytopes or Banach spaces,
- extension problems for polytopes or Banach spaces.

Lifting problems for polytopes

Basic example: does the convex map



$$\operatorname{conv}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

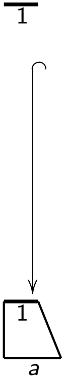
$$\Big\downarrow \; \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\operatorname{conv}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

have a convex section? (Answer: No.)

Extension problems for polytopes

Basic example: does the convex map



$$\operatorname{conv}\left\{\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\} \hookrightarrow \operatorname{conv}\left\{\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\a\end{pmatrix}\right\}, \quad \begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}x\\y\end{pmatrix}$$

have a convex retraction? (Answer: if and only if $a \leq 1$.)

Part 3: The metric aspect

A natural metric on a convex space

Under certain boundedness conditions, the assignment

$$d(x, y) = \sup_{f: C \to [0,1] \text{ convex}} |f(x) - f(y)|$$

defines a metric on C.

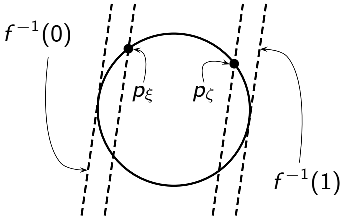
Example: quantum states

Let $\mathcal{H}$ be a Hilbert space. For $\xi \in \mathcal{H}$ with $||\xi|| = 1$, let p_ξ be the orthogonal projection operator onto the subspace spanned by ξ. Take

$$C = \mathrm{conv}\left\{ p_\xi,\, \xi \in \mathcal{H} \setminus \{0\} \right\}.$$

In quantum theory, C is the space of **states of a quantum system**.

$$d(p_\xi, p_\zeta) = \sqrt{1 - |\langle \xi, \zeta \rangle|^2}$$



The Wasserstein distance

The metric

$$d(x,y) = \sup_{f:C\to[0,1] \text{ convex}} |f(x) - f(y)|$$

resembles the Wasserstein distance of probability measures on a metric space (M,d):

$$d(\mu,\nu) = \sup_{\substack{f\,:\,M\to\mathbb{R} \\ |f(m)-f(m')|\le d(m,m')}} \left| \int f\,d\mu - \int f\,d\nu \right|$$

The comparison with noncommutative geometry

In noncommutative geometry, the Wasserstein distance

$$d(\mu,\nu) = \sup_{\substack{f\,:\,M\to\mathbb{R}\\ |f(m)-f(m')|\leq d(m,m')}} \left|\int f\,d\mu - \int f\,d\nu\right|$$

gets replaced by Connes' distance formula:

$$d(\mu,\nu) = \sup_{\substack{a\in\mathcal{A}\\ ||[D,a]||\leq 1}} |\mu(a)-\nu(a)|$$

Herby, $\mathcal{A}$ is a (noncommutative) algebra of operators on a Hilbert space $\mathcal{H}$, while D is an operator on $\mathcal{H}$ such that $[D,a]$ represents the derivative of a. μ and ν are positive functionals on $\mathcal{A}$.

Towards noncommutative geometry with convex spaces?

Idea

Think of a convex space C as being the space of probability measures over a (non-existing) geometry. Define distances via

$$d(\mu, \nu) = \sup_{\substack{f\,:\,C \to \mathbb{R} \text{ convex} \\ f \in L_1(C)}} |f(\mu) - f(\nu)|$$

where $L_1(C)$ is some convex subset of all the functionals $C \to \mathbb{R}$.
$L_1(C)$ symbolizes the set of Lipschitz continuous functions.

Chu spaces

Idea

Take $L_1(C)$ itself as the data defining the geometry!

Then there is a pairing

$$C \times L_1(C) \to \mathbb{R}, \quad (x, f) \mapsto f(x)$$

More generally, one can consider any pair of convex spaces (C, D) equipped with a pairing

$$\mathrm{ev} : C \times D \to \mathbb{R}, \quad (x, y) \mapsto \mathrm{ev}(x, y).$$

This is the convex version of a **Chu space**. The induced metric is

$$d(x, x') = \sup_{y \in D} |\mathrm{ev}(x, y) - \mathrm{ev}(x', y)|$$

Generalizes Connes' distance formula!

Summary

Conclusion

- Abstract convexity is the study of the intrinsic algebraic structure of convex sets.
- This setting contains an abstract algebraic Hahn-Banach extension theorem.
- This theorem might provide hints on how to obtain an abstract convex homotopy theory yielding algebraic invariants of convex spaces.
- Convex spaces could be an interesting framework for noncommutative metric geometry.

⇒ Many directions for future work!

Bibliography

- Tobias Fritz, *Convex Spaces I: Definition and Examples*, http://arxiv.org/abs/0903.5522.
- Paul Goerss, Kristen Schemmerhorn, *Model Categories and Simplicial Methods*, http://arxiv.org/abs/math.AT/0609537.
- Francesco D'Andrea, Pierre Martinetti, *A view on optimal transport from noncommutative geometry*, http://arxiv.org/abs/0906.1267.

Backup slides

Combinatorial convex spaces

Definition

A convex space C is said to be **combinatorial** whenever all convex combinations

$$\lambda x + \overline{\lambda} y$$

are independent of λ.

So a combinatorial convex space is a set C together with a single binary operation

$$\wedge : C \times C \longrightarrow C$$

which is idempotent, commutative and associative. By defining an order structure as

$$x \leq y \quad \Longleftrightarrow \quad x = x \wedge y,$$

it can be shown that such a C is nothing but a semilattice. (a partially ordered set where every two elements have a greatest lower bound.)

The classification of convex spaces

A generic convex space is neither geometric nor combinatorial.

Nonetheless, every convex space can be decomposed into a combinatorial part and a geometrical part:

Theorem (informal)

Every convex space C can be written as a bundle

$$C \xrightarrow{p} \widetilde{C}$$

where $\widetilde{C}$ is combinatorial and all fibers $p^{-1}(c)$ for $c \in \widetilde{C}$ are geometric.