CHALLENGES AND LIMITATIONS OF FUZZY DARK MATTER SIMULATIONS

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What is “fuzzy dark matter”?

- F(C)DM, BECDM, ULDM, ELBDM, ψDM, quantum-wave DM, (ultra-light) axion(-like) DM (ULA, ALP)...
- New **extremely light scalar** particle ($m \approx 10^{-22}\,\mathrm{eV}$!)
- Non-thermal production mechanism (thus not ultra-hot)
- Aggregations of bosons can form a **Bose–Einstein condensate**
- Quantum effects counteract gravity at **small scales** (uncertainty principle), erase structure
- Tiny mass
  $\Rightarrow$ large de Broglie wavelength ($\lambda \sim 1/m$)
  $\Rightarrow$ **macroscopic quantum effects** on kpc scales

Motivation for fuzzy dark matter

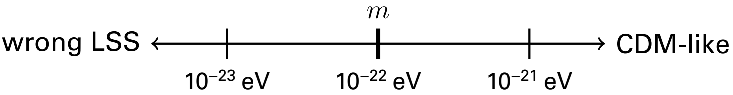
Particle physics perspective:

- Original concept – strong CP problem:
  Why doesn't QCD violate CP symmetry?
- Solved by Peccei–Quinn $\mathrm{U}(1)$ symmetry
  and (pseudo-)scalar field (*axion*!)
  Peccei and Quinn (1977)!
- Fuzzy dark matter is **not** the QCD axion, but axion-like particles are a common feature of early-universe theories

Astrophysics perspective:

- Small-scale challenges (cusp–core, missing satellites, …)

$\rightarrow$ Ultra-light scalars: WIMP alternative, could improve this



- **No sign of (WIMP) CDM**

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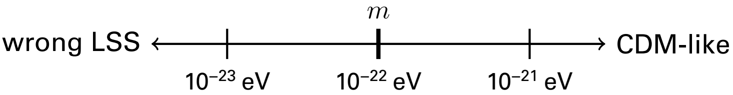

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Derivation of the fuzzy dark matter equations

- Start with simple scalar field action

$$S = \frac{1}{\hbar c^2} \int \mathrm{d}^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2} \frac{m^2 c^2}{\hbar^2} \phi^2 - \frac{\lambda}{\hbar^2 c^2} \phi^4 \right)$$

$\rightarrow$ superfluid DM without self-interaction ($\lambda = 0$ or $T \rightarrow 0$)

- Non-relativistic limit yields the Schrödinger equation

$$i\hbar \left(\partial_t \psi + \frac{3}{2} H \psi \right) = -\frac{\hbar^2}{2m} \nabla^2 \psi + m \Phi \psi$$

- Mean field approximation: interpretation as single macroscopic wave function of BE condensate with density $\rho = m|\psi|^2$
- "FDM equations": non-linear Schrödinger–Poisson system

$$i\hbar \partial_t \psi_\mathrm{c} = -\frac{\hbar^2}{2ma^2} \nabla_\mathrm{c}^2 \psi_\mathrm{c} + \frac{m}{a} \Phi_\mathrm{c} \psi_\mathrm{c}$$
$$\nabla_\mathrm{c}^2 \Phi_\mathrm{c} = 4\pi G m (|\psi_\mathrm{c}|^2 - \langle |\psi_\mathrm{c}|^2 \rangle)$$

Only a single scale, determined by $\frac{\hbar}{m}$ ($\rightarrow$ wavelength)

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Approaches to fuzzy dark matter simulations

I. Schrödinger–Poisson equations

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2ma^2}\nabla^2\psi + \frac{m}{a}\Phi\psi$$
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II. Madelung formulation (fluid dynamics representation)

$$\partial_t\rho + \nabla\cdot\rho\vec{v} = 0$$
$$\partial_t\vec{v} + \frac{1}{a^2}(\vec{v}\cdot\nabla)\vec{v} = -\nabla\left(\frac{1}{a}\Phi - \underbrace{\frac{\hbar^2}{2m^2a^2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}}_{=Q}\right)$$
$$\nabla^2\Phi = 4\pi G(\rho - \bar{\rho})$$

$$\psi = \sqrt{\frac{\rho}{m}}e^{i\alpha}$$
$$\rho = m|\psi|^2$$
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- Phase is undefined for $\rho = 0$
  $\Rightarrow$ significant effects on overall evolution

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“quantum potential”
“quantum pressure”

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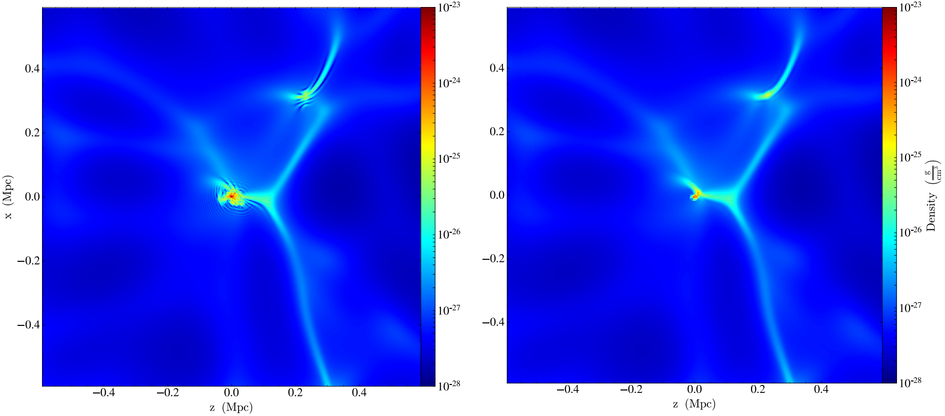
Schrödinger–Poisson vs. Madelung formulation



Li, Hui, and Bryan (2018)

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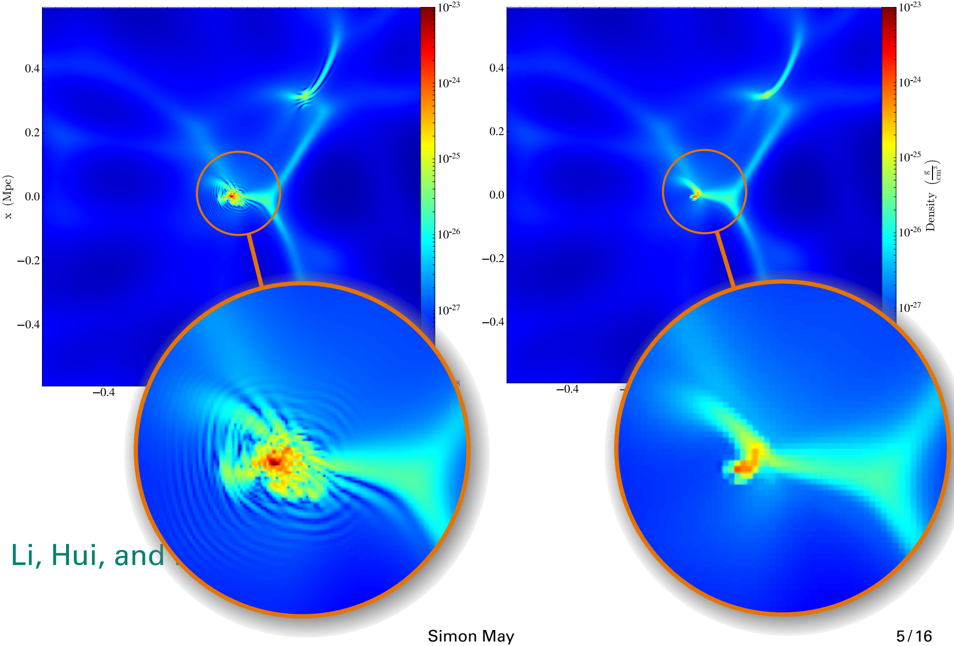

Li, Hui, and

Using pseudo-spectral methods to simulate FDM (in AREPO)

- Symmetrized split-step Fourier method ("kick–drift–kick")

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- Algorithm:

$$\psi \leftarrow e^{-i\frac{m}{\hbar}\frac{\Delta t}{2}\Phi}\psi \quad \text{kick}$$

$$\psi \leftarrow \mathrm{IFFT}\left(e^{-i\frac{\hbar}{m}\frac{\Delta t}{2}k^2}\,\mathrm{FFT}(\psi)\right) \quad \text{drift}$$

$$\Phi \leftarrow \mathrm{IFFT}\left(-\frac{1}{k^2}\,\mathrm{FFT}\left(4\pi Gm(|\psi|^2 - \langle|\psi|^2\rangle)\right)\right) \quad \text{update potential}$$

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Choice of time step: $\Delta t < \min\left(\frac{4}{9\pi}\frac{m}{\hbar}a^2\Delta x^2 \;,\; 2\pi\frac{\hbar}{m}a\frac{1}{|\Phi_{\max}|}\right)$

- "Exact" solution
- Automatic conservation of mass
- Can adapt existing PM code in AREPO
- Simple implementation

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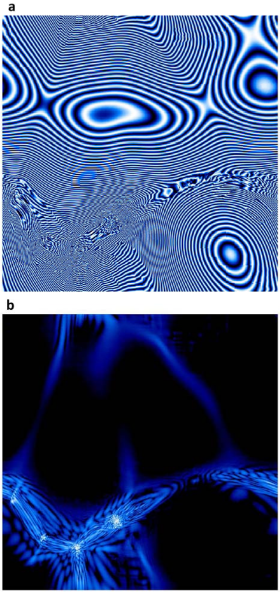
Why is it hard to simulate FDM?

Computational challenges

- Tiny mass $\leftrightarrow$ macroscopic quantum effects, de Broglie wavelength of galactic scale
- Both large scales and de Broglie scale must be resolved for correct evolution (sub-kpc cores can form)
- **Time step criterion:** $\Delta t \sim \Delta x^2$ (seems to be approach-independent)
- Tooling: Hydrodynamics codes are designed for N-body simulations



Schive, Chiueh, and Broadhurst (2014)

Correspondence of CDM and FDM initial conditions

Constructing a wave function ψ from a phase space distribution function f:

$$\psi(\vec{x}) \sim \sum_{\vec{v}} \sqrt{f(\vec{x}, \vec{v})} e^{im/\hbar\, \vec{x} \cdot \vec{v} + R_{\vec{v}}}$$

For “cold”/single-stream distribution function:

$$\psi = \sqrt{\frac{\rho}{m}} e^{i\alpha}$$

$$\vec{v} = \frac{\hbar}{m} \nabla \alpha$$

Grid discretization implies a maximum velocity which can be represented

Mocz, Lancaster, et al. (2018), Mocz, Fialkov, et al. (2019)

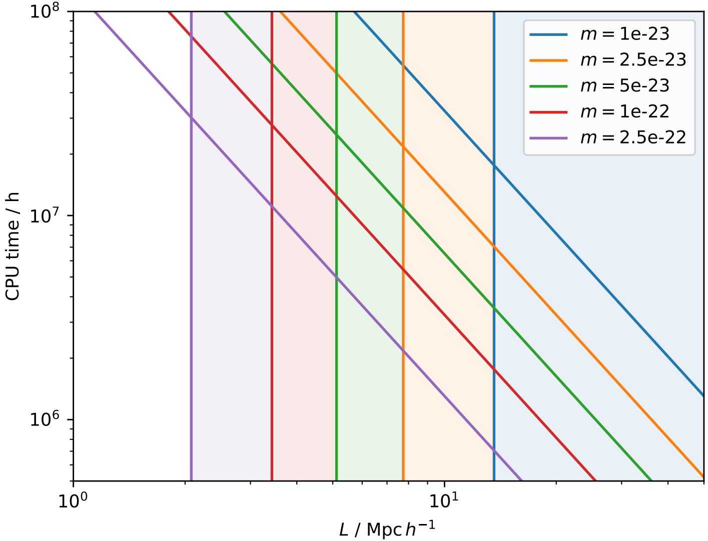
$$v < \frac{\hbar}{m} \frac{\pi}{\Delta x}$$

Cosmological box size limits ($z = 0$)

$$\Delta t < \frac{4}{9\pi}\frac{m}{\hbar}a^2\Delta x^2$$

$$v < \frac{\hbar}{m}\frac{\pi}{\Delta x}$$

Computational cost scales as $\sim N^5$ with grid size!

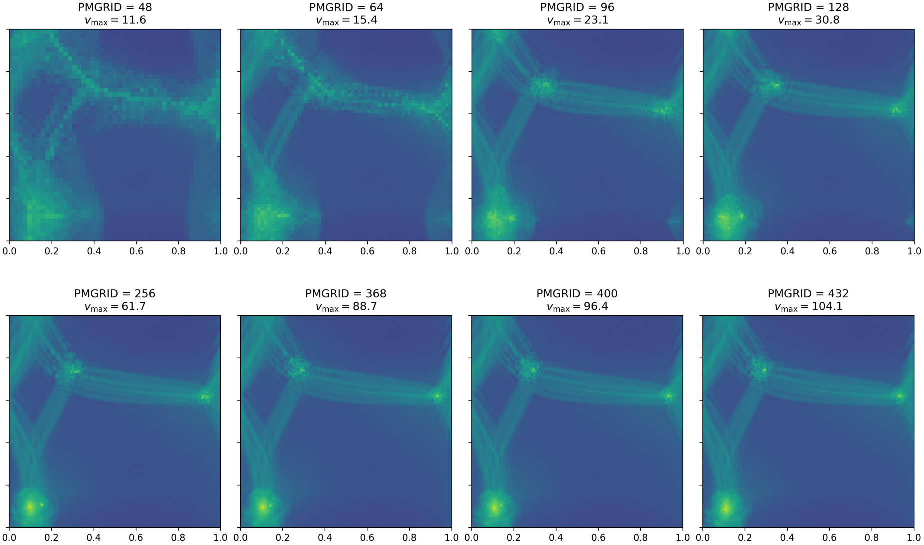


8192^3 grid

10^7 CPU h $\triangleq$ two full months of entire MPA cluster (FREYA)!

Cosmological 1 Mpc boxes in projection

$z = 0, m = 2.5 \times 10^{-23}\,\mathrm{eV}, v_{\Lambda\mathrm{CDM}} = 97.2\,\mathrm{km/s}$



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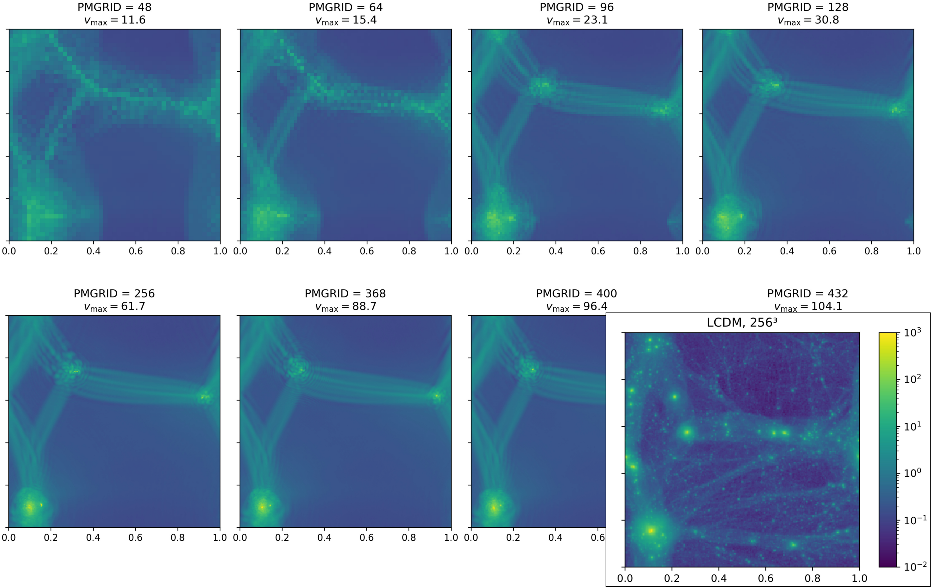
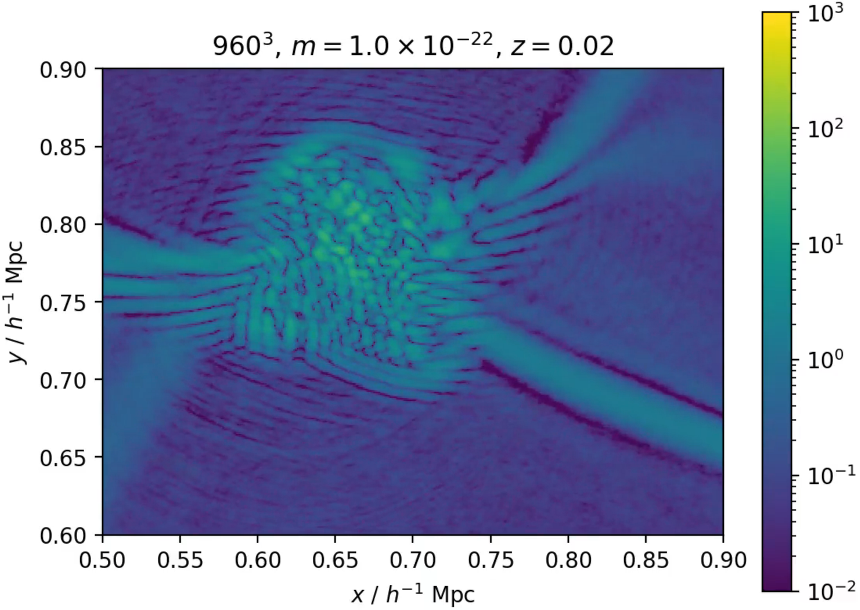
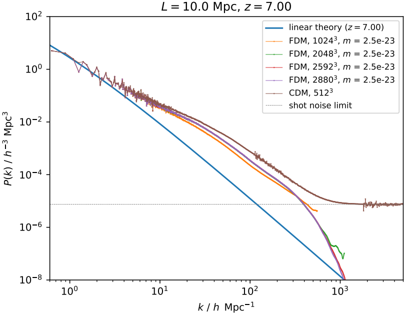


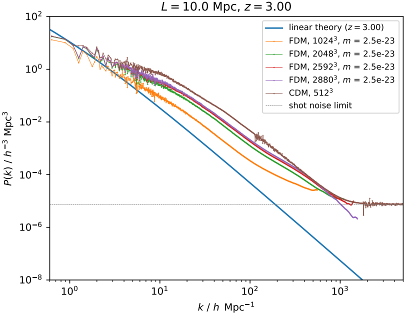
FDM movie (slice through halo, 1 Mpc box)

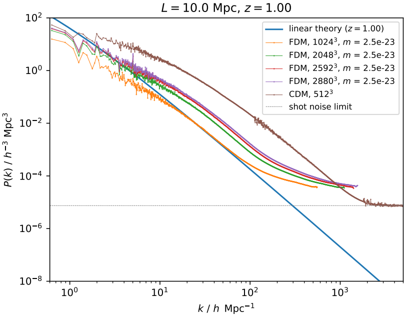


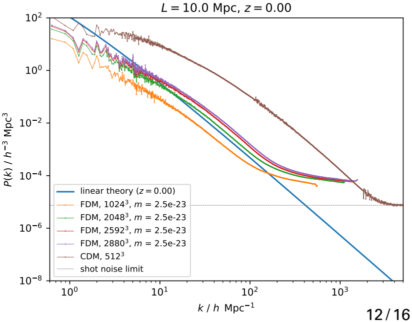
Matter power spectra

→ difficult to achieve converged resolution (≈ de Broglie wavelength)

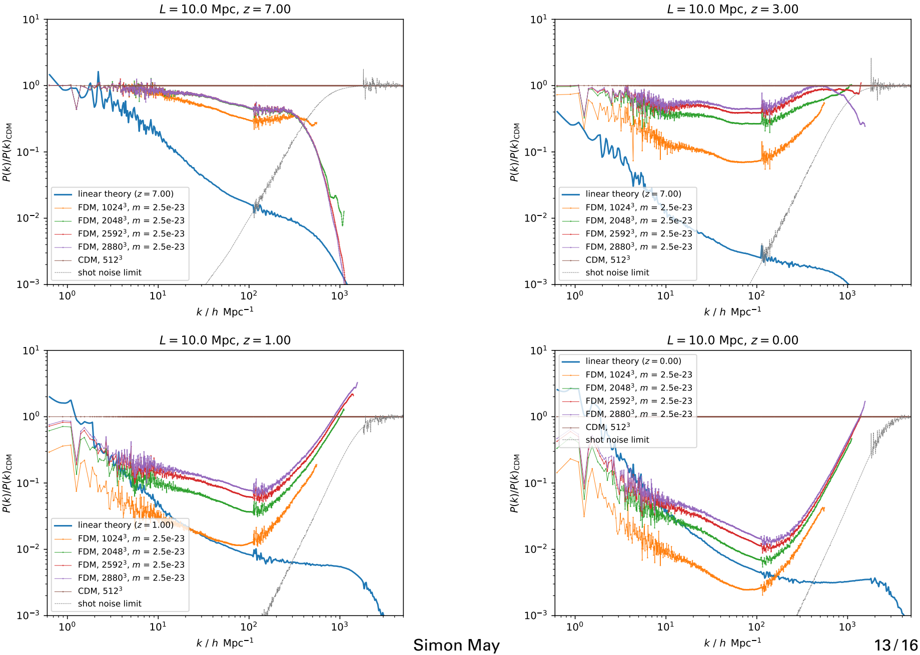




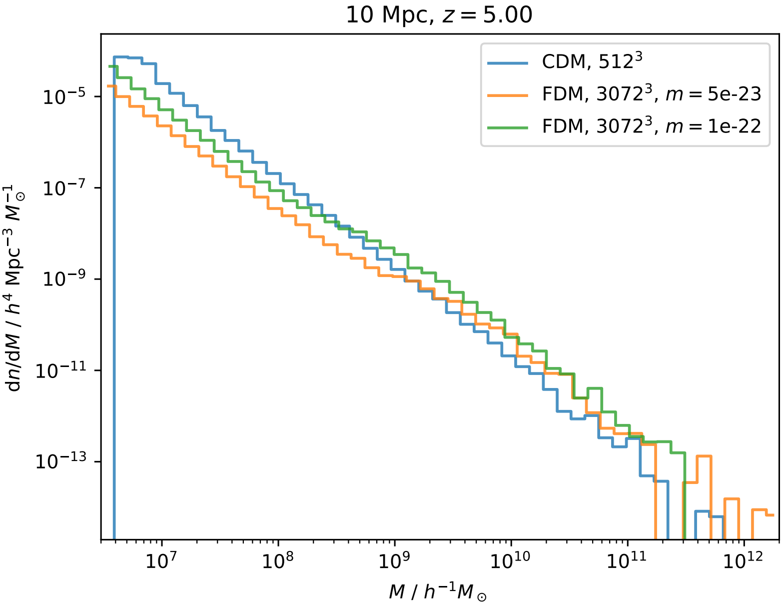




Matter power spectra relative to CDM



Halo finding: halo mass function



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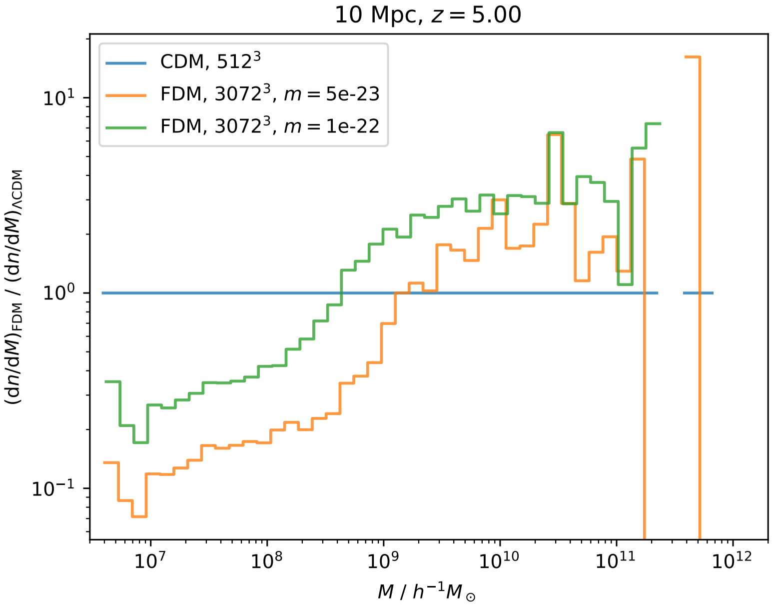

Short summary of the main problems for simulations

1. **Time integration** $\Delta t \sim \Delta x^2$
2. **Rapid oscillations** even in low-density regions since velocity corresponds to the gradient of the phase ($\rightarrow$ velocity criterion)
3. **Large dynamic range**: "large"-scale structure simulations still require resolving de Broglie wavelength
4. "New" field without decades of experience or refined codes/methods as for CDM

Currently running: 8640^3 simulation, 10 Mpc box ($\Delta x = 1.16$ kpc; for comparison: $\lambda_{\mathrm{dB}} = 1.21$ kpc for $m = 10^{-22}$ eV, $v = 100\,\mathrm{km/s}$)
$\rightarrow \approx 3 \times 10^6$ CPU h

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