

Spacetime Entanglement Entropy in 1+1 Dimensions

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ABSTRACT: Reference [1] defines an entropy for a gaussian scalar field ϕ in an arbitrary region of either a continuous spacetime or a causal set, given only the correlator $\langle\phi(x)\phi(y)\rangle$ within the region. The definition is global and independent of any choice of spacelike hypersurface. As a first application, we compute numerically the entanglement entropy in two cases where the asymptotic form is known or suspected from conformal field theory, finding excellent agreement when the required ultraviolet cut-off is implemented as a truncation on *spacetime* mode-sums. We also show how the symmetry of entanglement entropy reflects the fact that RS and SR share the same eigenvalues, R and S being arbitrary matrices.

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1 Introduction

It is customary to conceive of entropy in a quantum field theory as defined relative to a spacelike surface Σ on which the momentary state of the field is represented by a density-matrix $\rho(\Sigma)$. But for some purposes a more global notion of entropy would be preferable. For one thing, the notion of state at a moment of time might not survive in quantum gravity, and it seems in special jeopardy in relation to discrete theories, including causal sets [2] and others. Moreover, even in flat spacetimes, quantum fields are believed to be too singular to be meaningfully restricted to lower dimensional submanifolds, and in the context of quantum gravity with its fluctuating causal structure, this problem can only become worse. A more global conception of entropy is also called for if one aims at a path-integral or “histories-based” formulation of quantum mechanics. And such a conception would seem especially fitting in connection with black holes, whose very definition is global in character.

But over and above all these considerations stands the question of an ultraviolet “cutoff”. If one seeks to compute, for example, the entropy of entanglement of a scalar field between the interior and exterior of a black hole, one inevitably encounters a divergent answer that traces its existence to the infinitely many high frequency modes

of the field in the neighbourhood of the horizon. Within a particular Cauchy surface Σ , one can cut these modes off at some given wavelength λ but there is no guarantee that one would obtain the same answer if one tried to use the same cutoff with a different hypersurface. And without such a guarantee, it seems hard to feel fully confident in basic results like the proportionality of entanglement entropy to area [3, 4].

Thus arises the need for a covariant (locally Lorentz invariant) cutoff or — better still — a more fundamental theory of spacetime structure that would furnish nature’s own regularization scheme. Based on evidence from causal sets and such attempts as non-commutative geometry, one can anticipate that an entropy defined this way would need to refer to whole regions of spacetime rather than simply hypersurfaces. (For example, the spatio-temporal volume-element is invariant, but the spatial volume-element is not, a basic underpinning of causal set theory.) The need for a covariant discreteness or other covariant cutoff thus gives rise to a further need, the need for a definition of entropy that does not rely on the notion of state on a hypersurface.

Recently, an expression of this sort has been derived, which, for a gaussian scalar field (a free scalar field in a gaussian state), deduces an entropy for an arbitrary region R of spacetime from the correlation function of the field within that region, $\langle 0|\phi(x)\phi(x')|0\rangle$, where $|0\rangle$ is the given gaussian state [1]. When R is globally hyperbolic with Cauchy surface Σ , the resulting entropy can be identified with that of Σ , but unlike with previous formalizations of the entropy concept, this expression is *covariant* in the sense that it involves only space-time quantities.¹

An advantage of this method is that it is applicable to both continuum spacetimes and to discrete causal sets, where the fundamental discreteness provides naturally a frame-independent cutoff. In this paper we study the new entropy-expression in the continuum, in order to be able to compare its behaviour with known results about entanglement entropy arising in the context of conformal field theory (CFT).

Let us recall how entanglement entropy can be captured by the new formula. In conventional treatments, entropy is identified with the “Gibbs entropy”

$$S = -\text{Tr} \rho \ln \rho \tag{1.1}$$

of a density-matrix $\rho(\Sigma)$, where Σ is a hypersurface and ρ is a “statistical state” for this hypersurface. If Σ is divided into complementary subregions A and B , then the

¹This “covariant” entropy agrees formally with the usual one in situations where both can be defined, but it applies also to non-globally hyperbolic spacetime regions, to causal sets, and more generally to any algebra with bosonic generators, as illustrated by the quantum theories we study herein. (A fermionic analog also exists.) The new entropy is also new in the sense that it demands a different sort of UV cutoff, and each different way of introducing a cutoff is technically a different definition of entropy.

reduced density matrix for subregion A is

$$\rho_A = \text{Tr}_B \rho \tag{1.2}$$

and its entropy is

$$S_A = - \text{Tr} \rho_A \ln \rho_A . \tag{1.3}$$

Provided that the full density-matrix $\rho(\Sigma)$ is pure, one can refer to S_A (which then necessarily equals S_B) as the *entanglement entropy* between A and B . In the new approach, the entropy of A is taken to be that of the spacetime region $R_A = D(A)$, where $D(A)$ is the “causal development” or “domain of dependence” of A . (More generally, R_A can be any sub-region of $D(A)$ that contains A in its interior.) In spacetime language, S_A can be described as the entropy of entanglement between R and its so-called “causal complement”. And – modulo the usual caveats about S_A being infinite – it is a theorem that the new and old approaches produce the same result for S_A .

In what follows, we describe the spacetime entropy formula more fully and apply it to compute entanglement entropies in some cases of interest.

In Section 2 we review the derivation of the formula and point out that it yields correctly the thermal entropy of a simple harmonic oscillator, as shown in detail in Appendix B.

In Section 3 we consider a two-dimensional “causal diamond” immersed in the vacuum within a larger causal diamond (our choice of vacuum being described more fully in Appendix A). The spacetime entropy of the smaller diamond in this case measures (when interpreted spatially) the entanglement between a line-segment and its complement within a larger line-segment. In Section 4 we consider a similar case corresponding spatially to a segment embedded in a half-line. In both situations we carry out the computation numerically for a massless scalar field.

Finally, we devote Appendix C to an ab initio proof of the symmetry of entanglement entropy between a region and its complementary region. The pleasantly simple derivation given there relates this symmetry directly to the fact that the product of two matrices has the same eigenvalues, no matter in which order the matrices are multiplied.

2 The Entropy of a Gaussian Field

Let us review the derivation in [1]. We start by considering a single “degree of freedom” corresponding to a conjugate pair of variables q and p that satisfy $[q, p] = i$. From them we can form three independent correlators, $\langle qq \rangle$, $\langle pp \rangle$, and $\text{Re} \langle qp \rangle$. In a q -basis for this “degree of freedom”, a general Gaussian density matrix takes the form

$$\rho(q, q') \equiv \langle q | \rho | q' \rangle \propto \exp \left[-\frac{A}{2}(q^2 + q'^2) + \frac{iB}{2}(q^2 - q'^2) - \frac{C}{2}(q - q')^2 \right] , \tag{2.1}$$

where the real parameters A , B , and C are completely determined by the above correlators.

Given that the entropy, $S(\rho) = -\text{Tr} \rho \ln \rho$, has to be dimensionless and invariant under unitary transformations, it can only depend on the combination:

$$\langle qq \rangle \langle pp \rangle - (\text{Re} \langle qp \rangle)^2 = \frac{C}{2A} + \frac{1}{4} \quad (2.2)$$

In the original work on entanglement entropy in references [3, 4], it was shown that $S(\rho)$ takes the form

$$-S = \frac{\mu \ln \mu + (1 - \mu) \ln(1 - \mu)}{1 - \mu} \quad (2.3)$$

with

$$\mu = \frac{\sqrt{1 + 2C/A} - 1}{\sqrt{1 + 2C/A} + 1} \quad (2.4)$$

To express S directly in terms of the correlators, we can introduce the ‘‘Wightman’’ and ‘‘Pauli-Jordan’’ matrices,

$$W = \begin{pmatrix} \langle qq \rangle & \langle qp \rangle \\ \langle pq \rangle & \langle pp \rangle \end{pmatrix}$$

and

$$i\Delta = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} .$$

The matrix W corresponds in the field theory to $W(x, x') = \langle 0 | \phi(x) \phi(x') | 0 \rangle$, while Δ gives the imaginary part of W and corresponds to the commutator function defined by $i\Delta(x, x') = [\phi(x), \phi(x')]$. Then

$$S = (\sigma + 1/2) \ln(\sigma + 1/2) - (\sigma - 1/2) \ln(\sigma - 1/2), \quad (2.5)$$

where $\pm i\sigma$ are the eigenvalues of $\Delta^{-1}R$, with $R \equiv \text{Re}[W]$ being the (componentwise) real part of W . We can further simplify (2.5) by writing it in terms of the eigenvalues of $\Delta^{-1}W = \Delta^{-1}R + i/2$ rather than those of $\Delta^{-1}R$. Calling these eigenvalues $\pm i\omega_{\pm}$, we have $\pm i\omega_{\pm} = i(1/2 \pm \sigma)$, and our formula for the entropy becomes

$$S = \omega_+ \ln \omega_+ - \omega_- \ln \omega_- \quad (2.6)$$

where ω_+ and $-\omega_-$ are now the two solutions λ of the generalized eigenvalue problem:

$$W v = i\lambda \Delta v . \quad (2.7)$$

In terms of these eigenvalues (2.6) becomes simply

$$S = \sum \lambda \ln |\lambda| . \quad (2.8)$$

In the special case just treated, Δ was invertible and we could just as easily have written (2.7) as an eigenvalue equation for $\Delta^{-1}W$. In general, however, Δ will have “zero modes” and will not be invertible, which is why we wrote (2.7) in the way that we did. When Δ is not invertible we can still define λ via (2.7), but in solving it we will add the further proviso² that v must belong to the *image* of Δ . With this proviso, our formulas for a single degree of freedom remain valid for many degrees of freedom, and S is given by (2.8) with the sum taken over the full set of independent solutions of (2.7).

To fully justify this prescription and delineate its exceptional cases we would need to take a detour into operator algebras and irreducible representations in Hilbert space. This would lead to a more algebraic definition of entropy and a proof of its equivalence to (1.1) under the assumption that an irreducible representation exists. Finally, we would prove that this more general entropy was given by (2.8), extended to a sum over the full spectrum of eigenvalues λ . We would also need to analyze the further subtleties that arise when Δ has zero-modes which are not also zero-modes of R . For a fuller discussion of these points, we refer the reader to [1].

We have arrived at a formulation that, for a gaussian field, expresses the entropy directly in terms of the pairwise correlation functions of the theory. This formulation can be applied to fixed-time correlators of course,³ but it also allows us to work with the *spacetime* two-point function of the theory, its Wightman function. In subsequent sections we do this within two-dimensional “causal diamonds”. As a first check of our framework, however, one can examine the “0+1 dimensional” case of a harmonic oscillator at finite temperature. In Appendix B, we do so and confirm that the expected result is obtained.

As we have already mentioned, the more generally defined entropy of a region (2.8) can in certain cases be interpreted as an entanglement entropy. In the next two sections we consider two such examples in flat 2-dimensional spacetime. However, for a massless field, there exists no consistent vacuum in two-dimensional Minkowski space, \mathbb{M}^2 . For

²Instead of restricting v in this way, we could instead construe it as an equivalence-class of solutions, two solutions being equivalent when their difference is annihilated by Δ . This quotient construction is equivalent to limiting v to the image of Δ , but it is more “invariant”, because u and Δu belong to different vector spaces, whence neither W nor Δ can act on $v = \Delta u$ without the aid of an auxiliary metric (which for us will be the L^2 inner product). That the two ways of proceeding yield the same entropy in the end can be verified explicitly in the oscillator example of Appendix B.

³in which case it agrees with that of [5]

this reason, we will carry out the calculation of Section 3 in a larger causal diamond that serves as infrared cutoff. We then need to choose a vacuum for this larger diamond. Our choice will be based on a recently proposed distinguished ground state for a free scalar field theory in a globally hyperbolic region or spacetime. This “ground state” or “vacuum” is called the “SJ” (Sorkin-Johnston) vacuum, and its definition is reviewed in Appendix A.

For consistency in speaking of entanglement entropy, it is important that the global entropy vanish. When the Wightman function is the SJ one W_{SJ} , the entropy does in fact vanish, for the eigenvalues in (2.8) are by construction either $\lambda = 1$ or $\lambda = 0$. Each term in the sum (2.8) is therefore zero. This outcome was to be expected, since the SJ vacuum is a pure state.

3 Entanglement Entropy I: Small Diamond in Big Diamond

We apply the formalism described in the previous section to compute the entanglement entropy of a causal diamond embedded in a larger 1+1 dimensional causal diamond spacetime, as shown in Figure 1. A causal diamond (also called order-interval or Alexandrov neighborhood) is the intersection of the future of a point p with the past of a point $q \succ p$. As is evident in the figure, each diamond is the domain of dependence of the 1d interval that is its “waist” or “diameter”. Thus our result for the entropy of the smaller diamond should be compared with the CFT-results for the entanglement entropy between a shorter line-segment and a longer one containing it.

Usually periodic boundary conditions are imposed in the CFT calculations, and with this choice, the entanglement entropy for a massless scalar field has been found to take the asymptotic form for $a \rightarrow 0$ [6, 7] (see also [8]),

$$S \sim \frac{1}{3} \ln \left[\frac{\tilde{L}}{\pi a} \sin \left(\frac{\pi \tilde{\ell}}{\tilde{L}} \right) \right] + c, \quad (3.1)$$

where a is a UV cutoff, $\tilde{\ell}$ is the length of the shorter interval, \tilde{L} is the length of the longer interval, and c is a non-universal constant.⁴ In the limit that the smaller interval

⁴The entropy (3.1) is supposed to be defined within an overall vacuum state, which however doesn’t quite exist because a massless scalar field on a circle (spacetime cylinder) has a zero-mode, which acts like a free particle and as such possesses no normalizable ground state. One can take a limit in which its energy goes to zero, however, and in this limit its contribution to the entanglement entropy seems to diverge logarithmically. The entanglement entropy would then be infinite, even with a UV cutoff. Presumably the CFT formulas have in mind regulating the zero-mode, by a small mass or otherwise, and holding the regulator fixed while sending the UV cutoff a to zero.

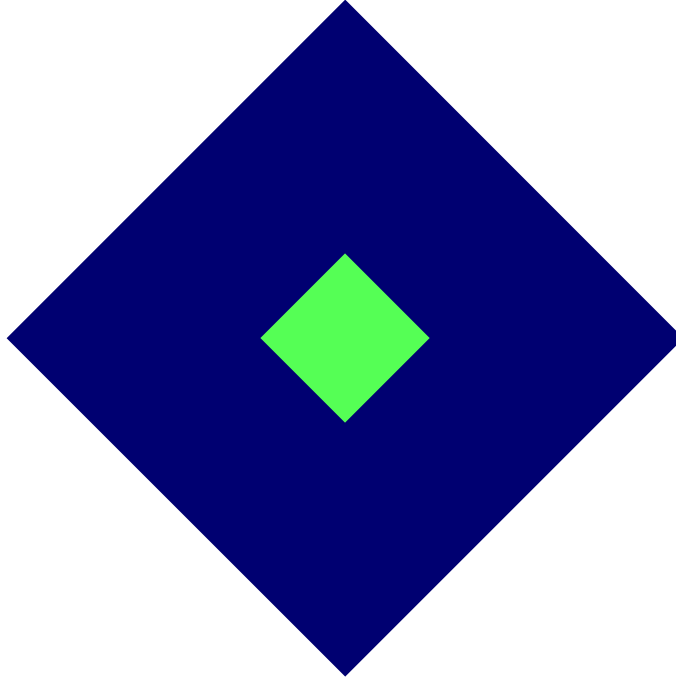


Figure 1. A diamond region within a diamond spacetime.

is much shorter than the larger one ($\frac{\tilde{\ell}}{L} \rightarrow 0$), the entropy reduces to

$$S \sim \frac{1}{3} \ln \left[\frac{\tilde{\ell}}{a} \right] + c . \quad (3.2)$$

In this limit, S depends only on the length of the smaller interval and the UV cutoff of the theory. For the massive theory, one would expect $1/m$ to play the role of IR scale, in which case the entropy, when $\tilde{\ell} \gg 1/m$, would take the form (cf. [6, 7]),

$$S \sim -\frac{1}{3} \ln[ma] \quad (3.3)$$

We now present a numerical calculation for the massless scalar field in the continuum. In setting it up we will borrow freely from reference [9], starting with the forms of $i\Delta$ and W for our model. In Minkowski lightcone coordinates $u = \frac{t+x}{\sqrt{2}}$ and $v = \frac{t-x}{\sqrt{2}}$, the Pauli-Jordan function is given by

$$\Delta(u, v; u', v') = \frac{-1}{2} [\theta(u - u') + \theta(v - v') - 1] . \quad (3.4)$$

For W , we will use the asymptotic form of W_{SJ} for a large causal diamond when the spacetime points of interest lie far from the corners of the diamond. In [9] this was found to be

$$W_{\text{centre}}(u, v; u', v') = -\frac{1}{4\pi} \ln|\Delta u \Delta v| - \frac{i}{4} \text{sgn}(\Delta u + \Delta v) \theta(\Delta u \Delta v) - \frac{1}{2\pi} \ln \frac{\pi}{4L} + \epsilon_{\text{centre}} + \mathcal{O}\left(\frac{\delta}{L}\right), \quad (3.5)$$

where $\epsilon_{\text{centre}} \approx -0.063$ and δ collectively denotes the coordinate differences $u - u', v - v', u - v', v - u'$. Here, $L = \tilde{L}/\sqrt{8}$ is the ‘‘half side length’’ of the larger diamond. (It will be convenient to work with L and its analog ℓ for the smaller diamond, rather than with the diameters $\tilde{\ell}$ and \tilde{L} ($\tilde{\ell} = 2\sqrt{2}\ell$ and $\tilde{L} = 2\sqrt{2}L$). We will also set $\ell \equiv 1$ and choose then $L = 100$ so that the $\ell/L = 0.01$ and we are in a regime where (3.2) applies.)

Within the smaller diamond the $\mathcal{O}\left(\frac{\delta}{L}\right)$ correction in (3.5) will be negligible, and we can write the remainder more simply as

$$W(u, v; u', v') = \lim_{\epsilon \rightarrow 0^+} \left(-\frac{1}{4\pi} \ln \left[-\mu^2 (\Delta u - i\epsilon)(\Delta v - i\epsilon) \right] \right) \quad (3.6)$$

where $\mu = (\pi/4L)e^{-2\pi\epsilon_{\text{centre}}}$ is the IR scale of the large diamond. As long as the small diamond is much smaller than the large one, this approximation should be adequate. In our calculation we will use (3.6) for W , with μ taken specifically to be $\mu = 0.0116681$.

We should note here that although the construction of W_{SJ} for a causal diamond is completely well-defined, it has no finite limit as the large diamond goes to infinity. Indeed a self-consistent Minkowski vacuum state $|0_M\rangle$ does not exist. If we try to define a vacuum in the usual way as the state annihilated by the operator coefficients of the positive frequency modes in the expansion of the field operator $\hat{\phi}(t, x)$, then we encounter an infrared divergence. We can remove the divergence by introducing a long wavelength cutoff into the integral for the Wightman function W , but the result is unphysical because it fails to be positive semidefinite as a quadratic form. Nevertheless, the resulting expression matches the general form (3.6) that we obtained as a local approximation to the SJ vacuum of the large diamond. In this sense, we can think of (3.6) as an approximate Minkowski vacuum which is valid for separations Δt and Δx that are small compared to the IR scale μ .

Returning to our calculation, we want to solve

$$Wv = i\lambda\Delta v \quad (3.7)$$

subject to

$$\Delta v \neq 0. \quad (3.8)$$

To that end we will represent W and Δ as matrices, using the basis that diagonalizes $i\Delta$, and which consists of two families of eigenfunctions:⁵

$$f_k(u, v) := e^{-iku} - e^{-ikv}, \quad \text{with } k = \frac{n\pi}{\ell}, \quad n = \pm 1, \pm 2, \dots \quad (3.9)$$

$$g_k(u, v) := e^{-iku} + e^{-ikv} - 2\cos(k\ell), \quad \text{with } k \in \mathcal{K}, \quad (3.10)$$

where $\mathcal{K} = \{k \in \mathbb{R} \mid \tan(k\ell) = 2k\ell \text{ and } k \neq 0\}$. The eigenvalues are $\lambda_k = \ell/k$. The L^2 -norms are $\|f_k\|^2 = 8\ell^2$ and $\|g_k\|^2 = 8\ell^2 - 16\ell^2\cos^2(k\ell)$.

Before actually embarking on the numerics, however, we need to decide on a cutoff. As we have been emphasizing, it will necessarily have a spacetime character as opposed to the purely spatial one seen, for example, in a lattice of carbon atoms. A discrete theory provides its own cutoff, but here in the continuum a naive lattice cutoff would be inconvenient and possibly inappropriate. Instead we simply truncate the matrices representing W and Δ by retaining only a finite number of eigenfunctions f_k and g_k up to a maximum value k_{max} of k . Finally, in comparing our results with (3.2), we need to translate our cutoff into a purely spatial one a . It is not certain that such a correspondence is always possible, but in this case we are expanding solutions of the wave equation, which in turn are in one-to-one correspondence with initial data specified on the spatial diameter of the causal diamond. With the modes we have retained, we can expand initial data of wavelengths longer than $\lambda_{min} \sim 1/k_{max}$ (or $2\sqrt{2}\pi/k_{max}$ if one were trying to be more precise). It is therefore natural to equate a to $1/k_{max}$, and this is what we do in the comparisons below.

In our basis, the integral-kernel $i\Delta$ is diagonal, so its representation is trivial, but for W , we must compute $\langle f_k|W|f_{k'} \rangle$ and $\langle g_k|W|g_{k'} \rangle$, which we did numerically.⁶ The terms $\langle f_k|W|g_{k'} \rangle$ vanish, making W block diagonal in this basis, so we can treat each block separately in solving (3.7). Summing over the resulting eigenvalues λ , we obtain the entropy associated to each block. Each block contributed to the entropy roughly equally, with the g block making a slightly greater contribution. Adding the two contributions, we obtain the total entropy. In the calculations reported here, all of the eigenvalues obtained from (3.7)-(3.8) were order-one numbers of absolute value below 3, with all but a handful of the eigenvalue-pairs being very close to the values one and zero. (As required for consistency we did not encounter any functions in the kernel of $i\Delta$.) The resulting entropies are plotted in Figure 2, as a function of $\frac{\ell}{a}$. As

⁵Thanks to (3.8) we need only consider functions orthogonal to the kernel of Δ , all of which consist of solutions to the wave equation. If one wanted to expand arbitrary L^2 functions, one would need to supplement the solutions, (3.9) and (3.10), with a basis for $\ker \Delta$.

⁶ We performed the calculations in this section and the next using Mathematica 9.0.

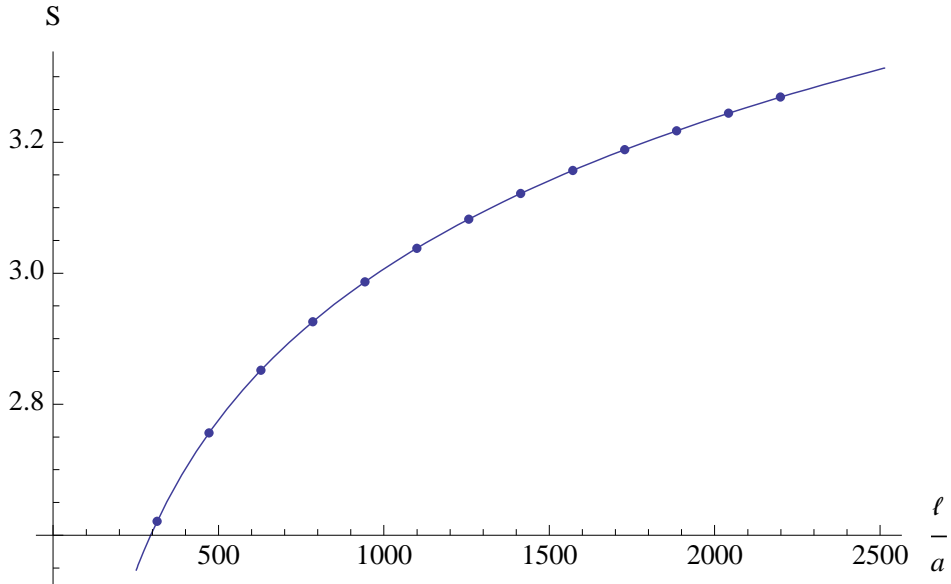


Figure 2. The entanglement entropy S versus ℓ/a . Data points represent calculated values of (2.8).

seen in the plot, the obtained values of S are fit almost perfectly by the curve

$$S = b \ln \left[\frac{\ell}{a} \right] + c \quad (3.11)$$

with $b = 0.33277$ and $c = 0.70782$. Thus, the entropies obtained from our “spacetime formulation” closely match the asymptotic form (3.2).

4 Entanglement Entropy II: Diamond in Halfspace

In the previous section, we calculated the entropy of a causal diamond in \mathbb{M}^2 , or more accurately a diamond embedded in the center of a much larger diamond. In this section, we do the analogous calculation for a causal diamond embedded in (and touching the boundary of) a spacetime equal to the right half of \mathbb{M}^2 . In terms of subregions of a spacelike hypersurface (or rather hyposurface) we are here computing entanglement entropy for a 1d interval at one end of a semi-infinite line, while in the previous section our interval was centered within a much larger but still finite interval.

Compared to full \mathbb{M}^2 , the half-space has the advantage that it admits a true minimum energy state or vacuum, relieving us of the need for an infrared cutoff (other than the cutoff always imposed by the electronic computer.) On the other hand, the

presence of a boundary requires that we choose a boundary condition. For the free massless scalar field which we consider, we will require the field to vanish at the boundary (“Dirichlet condition”), and our vacuum will be the ground state with respect to this condition. Of course the calculation itself cares nothing about the boundary, except indirectly insofar as it influences the Wightman function.

Our spacetime will comprise the subset of \mathbb{M}^2 defined by $t \in (-\infty, +\infty)$ and $x \in [0, +\infty)$. The diamond whose entropy we seek will be the one shown in Figure 3, its diameter being the interval $I = \{(t, x) \mid t = 0, x \in [0, 2\sqrt{2}\ell]\}$. As before, we will compute the entropy of a free massless scalar field $\phi(x)$. Under the boundary condition, $\phi(x = 0, t) = 0$, the solutions to the wave equation are

$$\mathcal{U}_k(x, t) = \sqrt{\frac{2}{|k|}} e^{-ikt} \sin(kx), \quad (4.1)$$

resulting in the following two point function for the vacuum state

$$W(x, t; x', t') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{dk}{|k|} e^{-i|k|(t-t')} \sin(kx) \sin(kx'). \quad (4.2)$$

One can explicitly check that $W(X, X') - W(X', X) = i\Delta(X, X')$ for $X, X' \in D(I)$, with Δ given by (3.4).

As before, we need to solve the eigenvalue problem,

$$Wv = i\lambda\Delta v, \quad (4.3)$$

where the integration is over the shaded region in Figure 3. (This would be the full domain of dependence of I , were we in all of \mathbb{M}^2 , but given the boundary it is only a subset thereof. The resulting loss of information for numerical purposes is compensated by the convenience of working within a rectangular shaped region.) Since $i\Delta$ is exactly the same operator as before, we will use the same basis-functions as in the previous section. (Notice however that the centre of the causal diamond is no longer the centre of the coordinate system.) The representation of Δ is trivial in this basis, while representing W requires us to compute $\langle f_k | W | f_{k'} \rangle$, $\langle f_k | W | g_{k'} \rangle$, and $\langle g_k | W | g_{k'} \rangle$. Calculating these inner products and introducing the same UV cut-off, a , we are able to evaluate the entanglement entropy. The result, as shown in Figure 4, is fit almost perfectly by

$$S = \frac{1}{6} \ln \left[\frac{\ell}{a} \right] + c, \quad (4.4)$$

with $c = 0.11465$. This again agrees with the CFT asymptotic form.⁷

⁷The coefficient is $1/6$ rather than $1/3$ because the entanglement concerns only one of the two boundaries of the smaller interval.

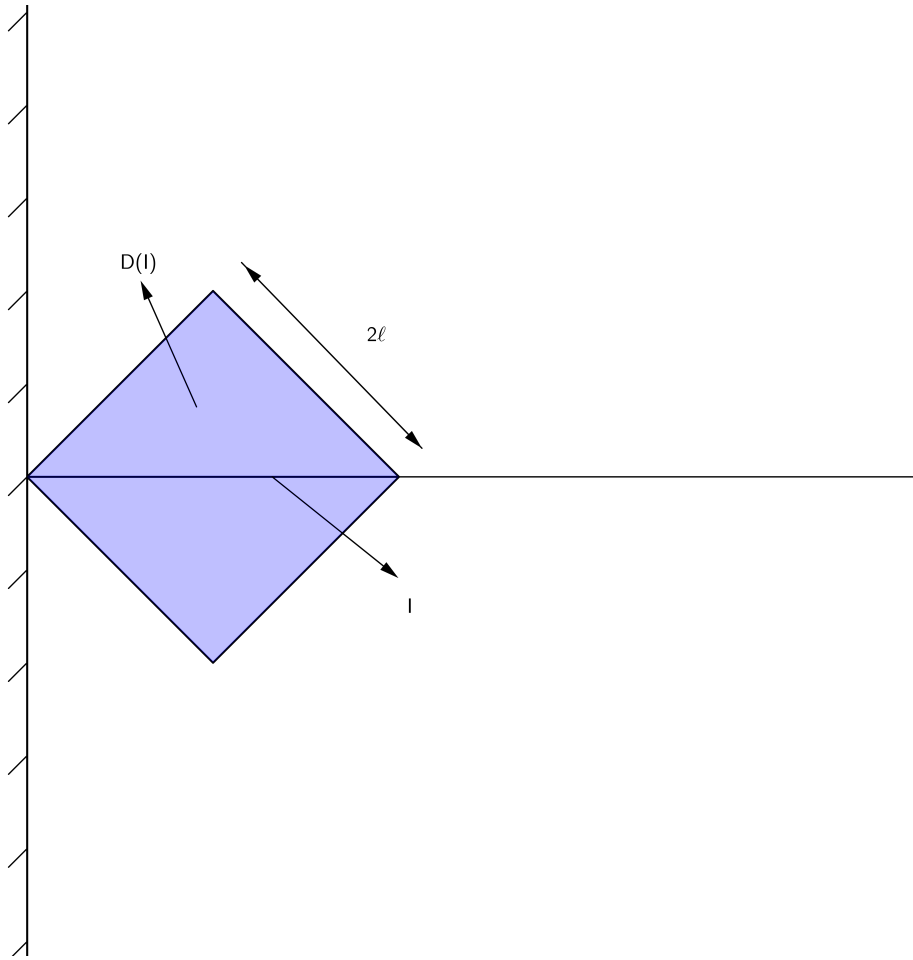


Figure 3. Spacetime diagram of the region of interest in this section.

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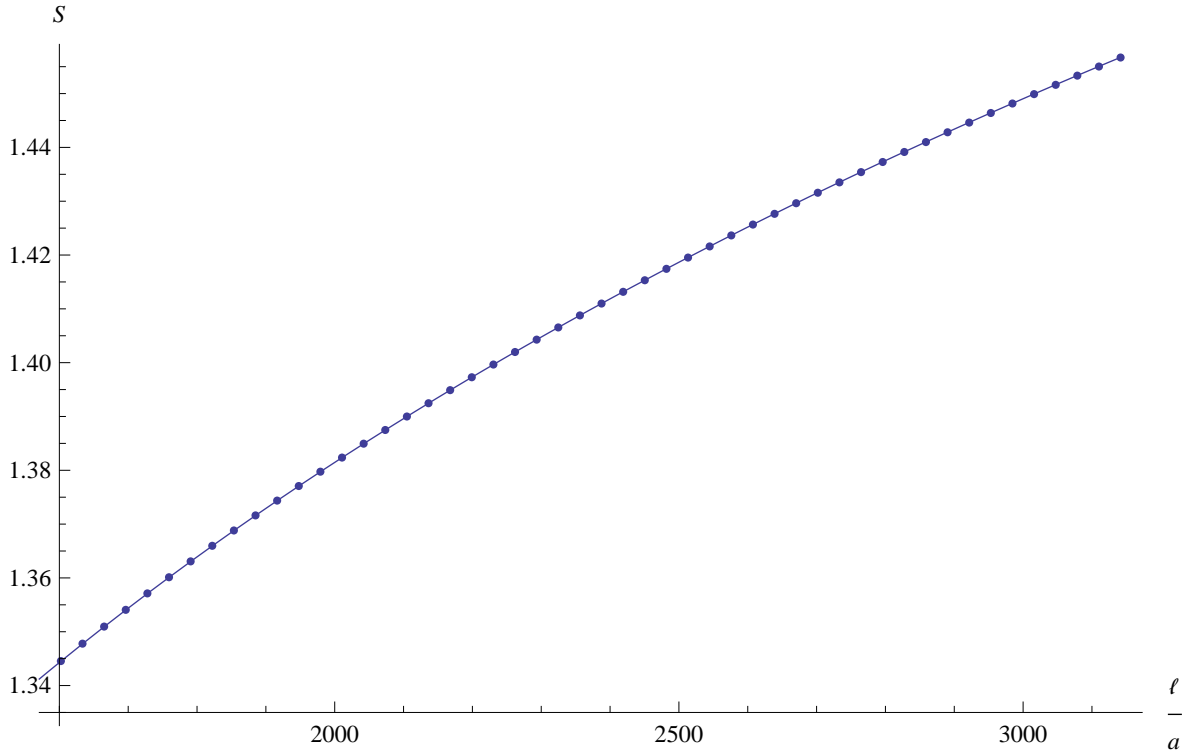


Figure 4. The entanglement entropy S versus the UV cutoff a for the case of the halfspace.

A The S-J Ground State

We review the SJ proposal for the ground state of a free scalar field in a d -dimensional globally hyperbolic spacetime \mathcal{M} with metric $g_{\mu\nu}$. [10, 11]. The starting point is the Pauli-Jordan function $\Delta(X, X')$, where X, X' denote spacetime points. It is defined by

$$\Delta(X, X') := G_R(X, X') - G_R(X', X) \quad (\text{A.1})$$

where G_R is the retarded Green function that satisfies $G_R(X, X') = 0$ unless $X' \prec X$, meaning that X' is to the causal past of X . The integral-kernel Δ is real and antisymmetric. It is related to the commutator by:

$$i\Delta(X, X') = [\hat{\phi}(X), \hat{\phi}(X')] . \quad (\text{A.2})$$

The Wightman function, or two-point function of a state $|0\rangle$, is

$$W_0(X, X') = \langle 0 | \hat{\phi}(X) \hat{\phi}(X') | 0 \rangle . \quad (\text{A.3})$$

The SJ vacuum is defined through its Wightman function by the three conditions [9]:

1. *commutator*: $i\Delta(X, X') = W(X, X') - W^*(X, X')$
2. *positivity*: $\int_{\mathcal{M}} dV \int_{\mathcal{M}} dV' f^*(X) W(X, X') f(X') \geq 0$
3. *orthogonal supports*: $\int_{\mathcal{M}} dV' W(X, X') W(X', X'')^* = 0,$

where $\int dV = \int d^d X \sqrt{-g(X)}$.

These conditions have the meaning that W is the positive part of $i\Delta$, thought of as an operator on the Hilbert space of square integrable functions $L^2(\mathcal{M}, dV)$ [10]. This allows us to describe a direct construction of W from the Pauli-Jordan function [11–13]. The distribution $i\Delta(X, X')$ defines the kernel of a (Hermitian) integral operator (which we may call the *Pauli-Jordan operator* $i\Delta$) on $L^2(\mathcal{M}, dV)$. The inner product on this space is

$$\langle f, g \rangle := \int_{\mathcal{M}} dV f(X)^* g(X), \quad (\text{A.4})$$

where $dV = \sqrt{-g(X)} d^d X$ is the invariant volume-element on \mathcal{M} . The action of $i\Delta : f \mapsto i\Delta f$ is then given by

$$(i\Delta f)(X) := \int_{\mathcal{M}} dV' i\Delta(X, X') f(X'). \quad (\text{A.5})$$

When this operator is self-adjoint, it admits a unique spectral decomposition. Whether or not this (or some suitable generalization of it) is the case depends on the functional form of the kernel $i\Delta(X, X')$ and on the geometry of \mathcal{M} . For the massless scalar field on a bounded region of Minkowski space, such as the finite causal diamond considered in this paper, $i\Delta$ is indeed a self-adjoint operator, since the kernel $i\Delta(X, X')$ is Hermitian and bounded. In the following we assume that we can expand $i\Delta$ in terms of its eigenfunctions.

Noting that the kernel $i\Delta$ is skew-symmetric, we find that the eigenfunctions in the image of $i\Delta$ come in complex conjugate pairs T_q^+ and T_q^- with real eigenvalues $\pm\lambda_q$:

$$(i\Delta T_q^\pm)(X) = \pm\lambda_q T_q^\pm(X), \quad (\text{A.6})$$

where $T_q^-(X) = [T_q^+(X)]^*$ and $\lambda_q > 0$. Now by the definition of $i\Delta$, these eigenfunctions must be solutions to the homogeneous Klein-Gordon equation. If they are L^2 -normalised so that $\|T_q^+\|^2 := \langle T_q^+, T_q^+ \rangle = 1$,⁸ then the spectral decomposition of the Pauli-Jordan operator implies that its kernel can be written as

$$i\Delta(X, X') = \sum_q \lambda_q T_q^+(X) T_q^+(X')^* - \sum_q \lambda_q T_q^-(X) T_q^-(X')^*. \quad (\text{A.7})$$

⁸ When the spacetime region has infinite volume, this must be replaced by a delta-function normalisation $\langle T_q^+, T_{q'}^+ \rangle = \delta(q - q')$.

We construct the SJ two-point function $W_{SJ}(X, X')$ by restricting (A.7) to its positive part:

$$W_{SJ}(X, X') := \sum_q \lambda_q T_q^+(X) T_q^+(X')^* = \sum_q \mathcal{T}_q(X) \mathcal{T}_q(X')^* \quad (\text{A.8})$$

where $\mathcal{T}_q(X) := T_q^+(X) \sqrt{\lambda_q}$.

B A Simple Illustration: Thermal Entropy of a Harmonic Oscillator

Let us apply the new entropy formula to the harmonic oscillator in one dimension. First we compute the entropy using the standard thermodynamic relation

$$S = \frac{\partial}{\partial T} (T \ln Z) \quad (\text{B.1})$$

For the harmonic oscillator, with energies $E_n = (n + \frac{1}{2})\omega$, the partition function is

$$Z = \frac{e^{-\frac{\beta\omega}{2}}}{1 - e^{-\beta\omega}} \quad (\text{B.2})$$

where as usual $\beta = 1/k_B T$ and we set $k_B \equiv 1$. From (B.2) and (B.1) we obtain the entropy as

$$S = -\ln[1 - e^{-\beta\omega}] + \frac{e^{-\frac{\beta\omega}{2}}}{1 - e^{-\beta\omega}}. \quad (\text{B.3})$$

Now we turn to computing the entropy using the new formula. With the field-operator $\hat{\phi}(t)$ identified as the oscillator's position-operator $\hat{q}(t)$ in the Heisenberg picture (and with the oscillator's mass set to unity or absorbed into q), we have for the commutator

$$i\Delta(t, t') = \frac{1}{2\omega} \left(e^{-i\omega(t-t')} - e^{i\omega(t-t')} \right), \quad (\text{B.4})$$

and for the thermal Wightman function,

$$W(t, t') = \frac{1}{2\omega} \left(\frac{e^{-i\omega(t-t')} + e^{i\omega(t-t')}}{e^{\beta\omega} - 1} + e^{-i\omega(t-t')} \right). \quad (\text{B.5})$$

In order to find the entropy, we need to solve the eigenvalue equation (2.7), which in the present context says

$$\int_L W(t, t') f(t') dt' = \lambda \int_L i\Delta(t, t') f(t') dt', \quad (\text{B.6})$$

the integration being over the interval L .

Since f is required to belong to the image of Δ , it must be a linear combination of $e^{\pm i\omega t}$, as is evident from (B.4). Writing then

$$A_{\pm} \equiv \int_L e^{\pm i\omega t} f(t) dt, \quad (\text{B.7})$$

we learn from (B.6) that

$$e^{-i\omega t} \left(\frac{e^{\beta\omega}}{e^{\beta\omega} - 1} - \lambda \right) A_+ = -e^{i\omega t} \left(\lambda + \frac{1}{e^{\beta\omega} - 1} \right) A_- . \quad (\text{B.8})$$

As this must hold for any value of $t \in L$, the coefficients of $e^{i\omega t}$ and $e^{-i\omega t}$ must be zero. Hence, we obtain two eigenvalues, each with multiplicity one,

$$\lambda = \frac{e^{\beta\omega}}{e^{\beta\omega} - 1} \quad \text{from } A_- = 0, A_+ \neq 0 \quad \text{and} \quad (\text{B.9})$$

$$\lambda = -\frac{1}{e^{\beta\omega} - 1} \quad \text{from } A_+ = 0, A_- \neq 0 . \quad (\text{B.10})$$

Substituting these two eigenvalues into

$$S = \sum \lambda \ln |\lambda| \quad (\text{B.11})$$

we obtain (B.3), the desired result.

C Symmetry of Entanglement from the Basic Formula (2.8)

Given a ‘‘bipartite’’ quantum system in an overall pure state, one knows that the entropies of the separate subsystems are necessarily equal. Here, after recalling the proof of this fact from the existence of Schmidt decompositions, we show how the same equality follows directly from our basic formula (2.8).

Let some Cauchy surface be divided into a subregion A and the complementary subregion B . Let

$$\rho_A = \text{Tr}_B \rho \quad (\text{C.1})$$

be the reduced density matrix for region A , and let

$$S_A = - \text{Tr} \rho_A \ln \rho_A \quad (\text{C.2})$$

be the entropy of this reduced density matrix. That one characterizes S_A as simply ‘‘the entanglement entropy’’, when the overall state of the field is pure, owes its consistency to the fact that it doesn’t matter which subregion one looks at: $S_A = S_B$.

Of course this equality is formal, since it relates two infinite quantities. However, in a finite-dimensional Hilbert space, it holds rigorously thanks to the Schmidt decomposition theorem: For any vector ψ_{AB} in a tensor product Hilbert space, there exist orthonormal sets $\{\psi_A\}$ and $\{\psi_B\}$ such that

$$\psi_{AB} = \sum_n \lambda_n^{1/2} \psi_A^{(n)} \psi_B^{(n)} \quad (\text{C.3})$$

where $\lambda_n^{1/2} > 0$ and $\sum_n \lambda_n = 1$. (This holds even if the Hilbert spaces \mathcal{H}_A and \mathcal{H}_B have different dimensions, in which case the index n in (C.3) cannot exceed the smaller of the two dimensions.) Now

$$\rho_A = \text{Tr}_B \psi_{AB} \psi_{AB}^\dagger = \sum_n \lambda_n \psi_A^{(n)} \psi_A^{(n)\dagger} \quad (\text{C.4})$$

and

$$\rho_B = \text{Tr}_A \psi_{AB} \psi_{AB}^\dagger = \sum_n \lambda_n \psi_B^{(n)} \psi_B^{(n)\dagger} \quad (\text{C.5})$$

Therefore, since ρ_A and ρ_B share the same nonzero eigenvalues λ_n , it follows that $S_A = S_B$.

We now prove this basic property of entanglement, using the new formulation. Let us divide the spacetime as shown in Figure 5, where 1 and 2 are causally disjoint globally hyperbolic regions whose union contains the whole spacetime in its “domain of dependence” (the union contains a Cauchy surface for the full spacetime). Restricting, now, the two-point functions, W and Δ , to the union of the two regions, we can write them in block-matrix form as follows, where the zeroes in Δ express the vanishing of the commutator at spatial separations.

$$\Delta = \begin{pmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{pmatrix}$$

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$$

For simplicity, let us now pretend that Δ is invertible, and define $M \equiv \Delta^{-1}W$. Because the overall state is pure by assumption, we know that M must have eigenvalues 0 and 1 and no others. (Otherwise the entropy would not vanish.) As an operator equation this says $M^2 = M$, which in turn yields when written out fully the two equations,

$$(\Delta_{11}^{-1}W_{11})^2 - \Delta_{11}^{-1}W_{11} = \Delta_{11}^{-1}W_{12}\Delta_{22}^{-1}W_{21} \quad (\text{C.6})$$

$$(\Delta_{22}^{-1}W_{22})^2 - \Delta_{22}^{-1}W_{22} = \Delta_{22}^{-1}W_{21}\Delta_{11}^{-1}W_{12} \quad (\text{C.7})$$

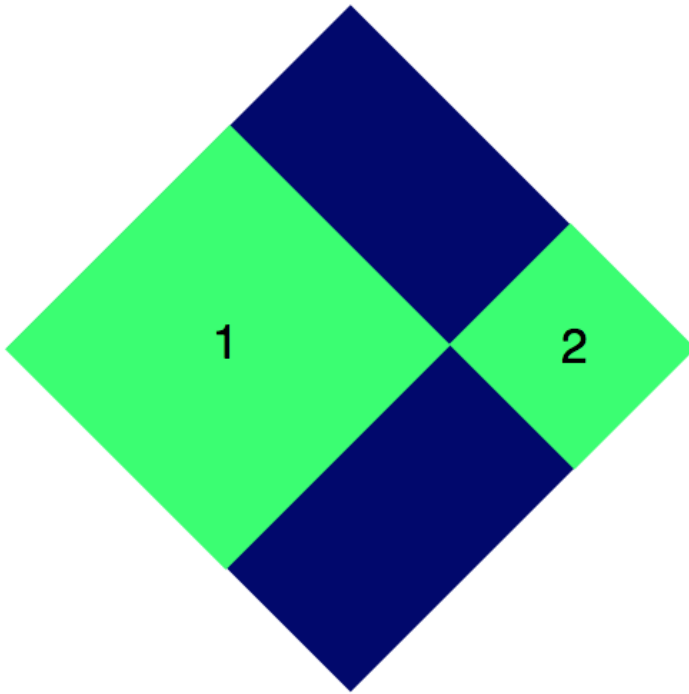


Figure 5. Division of a spacetime by two globally hyperbolic regions, 1 and 2.

This pair of equations has the form, $Q_1^2 - Q_1 = RS$ and $Q_2^2 - Q_2 = SR$, where $Q_1 = \Delta_{11}^{-1}W_{11}$ and $Q_2 = \Delta_{22}^{-1}W_{22}$. Using now the general fact⁹ that the nonzero spectrum of the product of two matrices is independent of the order in which the product is taken (this includes the multiplicity of the eigenvalues), we can conclude that $RS = Q_1^2 - Q_1$ and $SR = Q_2^2 - Q_2$ share the same nonzero eigenvalues:

$$\lambda_1^2 - \lambda_1 = \lambda_2^2 - \lambda_2 \quad \implies \quad (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - 1) = 0 \quad (\text{C.8})$$

For the eigenvalues λ that figure in equation (2.8), this means

$$\begin{cases} \lambda_1 = \lambda_2 \\ \lambda_1' = 1 - \lambda_2 = \lambda_2' \end{cases} \quad (\text{C.9})$$

⁹The nonzero spectrum of a finite-dimensional matrix M can be deduced directly from the traces of its powers, $\text{Tr}(M^n)$. (More precisely, one can deduce the multiset of its eigenvalues.) But cyclicity of the trace implies that for all n , $\text{Tr}[(RS)^n] = \text{Tr}[(SR)^n]$. Hence the matrix products $M = RS$ and $M = SR$ share the same nonzero spectrum. Notice that in our situation, the matrices R and S are not necessarily square because regions 1 and 2 are not necessarily of the same size.

where the last line follows from the fact that the eigenvalues of $\Delta_i^{-1}W_i$ come in pairs, λ and $\lambda' = 1 - \lambda$. Therefore $\Delta_{11}^{-1}W_{11}$ and $\Delta_{22}^{-1}W_{22}$ have the same (nonzero) spectrum, and $S_1 = S_2$.

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