TWO-SITE QUANTUM RANDOM WALK

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Abstract

We study the measure theory of a two-site quantum random walk. The truncated decoherence functional defines a quantum measure μ_n on the space of n-paths, and the μ_n in turn induce a quantum measure μ on the cylinder sets within the space Ω of untruncated paths. Although μ cannot be extended to a continuous quantum measure on the full σ-algebra generated by the cylinder sets, an important question is whether it can be extended to sufficiently many physically relevant subsets of Ω in a systematic way. We begin an investigation of this problem by showing that μ can be extended to a quantum measure on a "quadratic algebra" of subsets of Ω that properly contains the cylinder sets. We also present a new characterization of the quantum integral on the n-path space.

1 Introduction

A two-site quantum random walk is a process that describes the motion of a quantum particle that occupies one of two sites 0 and 1. We assume that the particle begins at site 0 at time $t = 0$ and either remains at its present site or moves to the other site at discrete time steps $t = 0, 1, 2, \ldots$. The transition amplitude is given by the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

Thus, the amplitude that the particle remains at its present position at one time step is $1/\sqrt{2}$ and the amplitude that it changes positions at one time step is $i/\sqrt{2}$. We can also interpret this process as a quantum coin for which 0 and 1 are replaced by T (tails) and H (heads), respectively.

This two-site process is a special case of a finite unitary system [5] in which more than two sites are considered. Although we present a special case we study the process in much greater detail, which we believe gives more insight into the situation. We expect that some of the work presented here will generalize to finite unitary systems. Moreover the methods employed will be general enough to cover non-unitary processes, which are ubiquitous for open systems and which plausibly include the case of quantum gravity (cf. [11]). We believe this greater generality is instructive, although it does lengthen the derivations in some instances.

Unlike previous studies of quantum random walks, the present work emphasizes aspects of quantum measure theory [3, 6, 7, 11, 12, 13]. We begin by introducing the n-truncated decoherence functional D_n on the n-path space Ω_n corresponding to U. The functional D_n is then employed to define a quantum measure μ_n on the events in Ω_n. We then use μ_n to define a quantum measure μ on the algebra of cylinder sets $\mathcal{C}(\Omega)$ for the path space Ω. Although μ cannot be extended to a continuous quantum measure on the σ-algebra generated by $\mathcal{C}(\Omega)$ [5], an important problem is whether μ can be extended to other physically relevant sets in a systematic way. We begin an investigation of this problem by introducing the concept of a quadratic algebra of sets. It is shown that μ extends to a quantum measure on a quadratic algebra that properly contains $\mathcal{C}(\Omega)$.

We also consider a quantum integral with respect to μ_n of random variables (real-valued functions) on Ω_n [6]. A new characterization of the quantum integral $\int f d\mu_n$ is presented. It is shown that a random variable $f\colon \Omega_n \to \mathbb{R}$ corresponds to a self-adjoint operator $\widehat{f}$ on a 2^n-dimensional Hilbert space H_n such that

$$\int f d\mu_n = \operatorname{tr}(\widehat{f} D^n)$$

where D^n is a density operator on H_n corresponding to D_n.

2 Truncated Decoherence Functional

The sample space (or "history-space") Ω consists of all sequences of zeros and ones beginning with zero. For example $\omega \in \Omega$ with

$$\omega = 0110110 \cdots$$

We call the elements of Ω *paths*. A finite string

$$\omega = \alpha_0 \alpha_1 \cdots \alpha_n,\ \alpha_i \in \{0,1\},\ \alpha_0 = 0$$

is called an *n-path*. If

$$\omega' = \alpha'_0 \alpha'_1 \cdots \alpha'_n$$

is another n-path, the *joint amplitude* or "Schwinger amplitude" $D^n(\omega, \omega')$ between ω and ω' is

$$D^n(\omega, \omega') = \tfrac{1}{2^n}\, i^{|\alpha_1 - \alpha_0|} \cdots i^{|\alpha_n - \alpha_{n-1}|} i^{-\left|\alpha'_1 - \alpha'_0\right|} \cdots i^{-\left|\alpha'_n - \alpha'_{n-1}\right|} \delta_{\alpha_n \alpha'_n} \tag{2.1}$$

We call the set Ω_n of n-paths the *n-path space* and write

$$\Omega_n = \{\omega_0, \omega_1, \ldots, \omega_{2^n - 1}\}$$

where $\omega_0 = 0 \cdots 0$, $\omega_1 = 0 \cdots 01$, $\omega_2 = 0 \cdots 010$, …, $\omega_{2^n-1} = 011 \cdots 1$. Thus, $\omega_i = i$ in binary notation, $i = 0, 1, \ldots, 2^n - 1$ and we can write $\Omega_n = \{0, 1, \ldots, 2^n - 1\}$. The *n-truncated decoherence matrix* (or *n-decoherence matrix*, for short) is the $2^n \times 2^n$ matrix D^n given by

$$D^n_{ij} = D^n(\omega_i, \omega_j) = D^n(i,j)$$

The algebra of subsets of Ω_n is denoted by $\mathcal{A}_n$ or 2^{Ω_n}. The *n-decoherence functional* $D_n\colon \mathcal{A}_n \times \mathcal{A}_n \to \mathbb{C}$ is defined by

$$D_n(A,B) = \sum \left\{ D^n_{ij} \colon \omega_i \in A, \omega_j \in B \right\}$$

The *n-truncated q-measure* $\mu_n\colon \mathcal{A}_n \to \mathbb{R}^+$ is defined by $\mu_n(A) = D_n(A, A)$ (we shall shortly show that $\mu_n(A) \geq 0$ for all $A \in \mathcal{A}_n$). We then have

$$\mu_n(A) = \sum \left\{ D_{ij}^n \colon \omega_i, \omega_j \in A \right\}$$

For $i, j = 0, 1, \ldots, 2^n - 1$ $i \neq j$ we define the *interference term*

$$I_{ij}^n = \mu_n\left(\{\omega_i, \omega_j\}\right) - \mu_n(\omega_i) - \mu_n(\omega_j)$$

Since

$$\begin{aligned} \mu_n\left(\{\omega_i, \omega_j\}\right) &= D_n\left(\{\omega_i, \omega_j\}, \{\omega_i, \omega_j\}\right) = D_{ii}^n + D_{jj}^n + 2\mathrm{Re}\, D_{ij}^n \\ &= \mu_n(\omega_i) + \mu_n(\omega_j) + 2\mathrm{Re}\, D_{ij}^n \end{aligned}$$

we have that

$$I_{ij}^n = 2\mathrm{Re}\, D_{i,j}^n = D_n(\omega_i, \omega_j) + D_n(\omega_j, \omega_i)$$

If $I_{ij}^n = 0$ we say that i and j *do not interfere*[1] and we write $i\, n\, j$; if $I_{ij}^n > 0$, then i and j *interfere constructively* and we write $i\, c\, j$; if $I_{ij}^n < 0$, then i and j *interfere destructively* and we write $i\, d\, j$.

Example 1. For $n = 1$, $\Omega_1 = \{00, 01\}$ and

$$D^1 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We have $\mu_1(\emptyset) = 0$, $\mu_1(\omega_0) = \mu_1(\omega_1) = 1/2$, $\mu_1(\Omega_1) = 1$. There is no interference and μ_1 is a measure

Example 2. For $n = 1$, $\Omega_2 = \{000, 001, 010, 011\} = \{0, 1, 2, 3\}$ and

$$D^2 = \frac{1}{4}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

We have $\mu_1(\emptyset) = 0$, $\mu_2(i) = 1/4$, $i = 0, 1, 2, 3$, $\mu\left(\{0, 2\}\right) = 0$

$$\begin{aligned} \mu_2\left(\{0,1\}\right) = \mu_2\left(\{0,3\}\right) = \mu_2\left(\{1,2\}\right) = \mu_2\left(\{2,3\}\right) = 1/2 \\ \mu_2\left(\{1,3\}\right) = 1,\ \mu_2\left(\{0,1,2\}\right) = 1/4 \\ \mu_2\left(\{0,1,3\}\right) = \mu_2\left(\{1,2,3\}\right) = 5/4,\ \mu_2(\Omega_2) = 1 \end{aligned}$$

[1] In some contexts, the stronger condition $D_n(\omega_i, \omega_j) = 0$ would be more appropriate.

In this case there is interference and μ_2 is not a measure. The interference terms are $I_{02}^2 = -1/2$, $I_{13}^2 = 1/2$ and $I_{ij}^2 = 0$ for $i < j$, $(i,j) \neq (0,2),(1,3)$.

Let $c_n(\omega)$ be the number of position changes for an n-path ω. For example, $c_4(01011) = 3$ and $c_5(011010) = 4$. If ω, ω' are n-paths it follows from (2.1) that

$$D^n(\omega,\omega') = \tfrac{1}{2^n}\, i^{[c_n(\omega)-c_n(\omega')]}\delta_{\alpha_n\alpha'_n} \tag{2.2}$$

If two integers are both even or both odd, they have the *same parity* and otherwise they have *different parity*. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $n \times x$ matrices, their *Hadamard product* $A \circ B$ is $[a_{ij}b_{ij}]$; that is, the ij-entry of $A \circ B$ is $a_{ij}b_{ij}$.

Theorem 2.1. *If D^n is the n-truncated decoherence matrix, then D^n is positive semi-definite, $D_{jk}^n = 0$ if j,k have different parity and if j,k have the same parity then $D_{jk}^n = 1/2^n$ when $c_n(j) = c_n(k) \pmod 4$ and $D_{jk}^n = -1/2^n$ when $c_n(j) \neq c_n(k) \pmod 4$. Moreover, $\sum_{j,k} D_{jk}^n = 1$.*

Proof. It is well-known that the Hadamard product of two positive semi-definite square matrices of the same size is again positive semi-definite. It follows from (2.2) that

$$D_{jk}^n = \tfrac{1}{2^n}\, i^{[c_n(j)-c_n(k)]}p_{jk} \tag{2.3}$$

where $p_{jk} = 1$ if j,k have the same parity and $p_{jk} = 0$, otherwise. Defining the matrices $P = [p_{jk}]$ and

$$C = \left[i^{[c_n(j)-c_n(k)]}\right]$$

we have that $D^n = \frac{1}{2^n}\, C \circ P$. Now C is clearly positive semi-definite. To show that P is positive semi-definite, notice that

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ & \vdots & & & & & \\ 0 & 1 & 0 & 1 & \cdots & 0 & 1 \end{bmatrix}$$

We see that P is self-adjoint, $\operatorname{rank}(P) = 2$ and $\operatorname{range}(P)$ is generated by the vectors $v_1 = (1,0,1,0,\ldots,1,0)$ and $v_2 = (0,1,0,1,\ldots,0,1)$. Now $Pv_1 =$

$2^{n-1}v_1$ and $Pv_2 = 2^{n-1}v_2$. For any 2^n-dimensional vector v with $v \perp v_1$ and $v \perp v_2$, we have that $Pv = 0$. Hence, the eigenvalues of P are 0 and 2^{n-1}. It follows that P is positive semi-definite and hence, D^n is positive semi-definite. The values of D^n_{jk} given in the statement of the theorem follow from (2.3). By symmetry, there are as many 1s as -1s among the off-diagonal entries of D^n. Hence,

$$\sum_{j,k=0}^{2^n-1} D^n_{jk} = \sum_{j=0}^{2^n-1} D^n_{jj} = \sum_{j=0}^{2^n-1} \frac{1}{2^n} = 1 \qquad \square$$

It follows from Theorem 2.1 that D^n is a density matrix.

Example 3. For $n = 3$ we have $\Omega_3 = \{0, 1, \ldots, 7\}$ and using vector notation $c_3 = (c_3(0), \ldots, c_3(7))$ we have that

$$c_3 = (0, 1, 2, 1, 2, 3, 2, 1)$$

Applying Theorem 2.1 we can read off the entries of D^3 to obtain

$$D^3 = \frac{1}{8}\begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Corollary 2.2. (a) *$inj, jnk \Rightarrow i\not{n}k$; inj, jdk or $jck \Rightarrow ink$.*
(b) *$icj, jck \Rightarrow ick$.* (c) *$idj, jdk \Rightarrow ick$.* (d) *$icj, jdk \Rightarrow idk$.*

Example 4. Referring to D^3 in Example 3 we see that $0d2$ $2c4$ and $0d4$. Also, $2d0$, $0d4$ and $2c4$. Finally, $1c3$, $3c7$ and $1c7$.

We now describe Hilbert space representations for D_n. Let H be a finite-dimensional complex Hilbert space. A map $\mathcal{E}\colon \mathcal{A}_n \to H$ satisfying $\mathcal{E}(\cup A_i) = \sum \mathcal{E}(A_i)$ for any sequence of mutually disjoint sets $A_i \in \mathcal{A}_n$ is a *vector-valued measure* on $\mathcal{A}_n$. If $\operatorname{span}\{\mathcal{E}(A)\colon A \in \mathcal{A}_n\} = H$, then $\mathcal{E}$ is a *spanning* vector-valued measure. The next result follows from Theorem 2.3 of [10] (cf. [4]).

Theorem 2.3. *Let D_n be the n-decoherence functional. There exists a spanning vector-valued measure $\mathcal{E}\colon \mathcal{A}_n \to \mathbb{C}^2$ such that $D_n(A,B) = \langle \mathcal{E}(A), \mathcal{E}(B)\rangle$ for all $A, B \in \mathcal{A}_n$. If $\mathcal{E}'\colon \mathcal{A}_n \to H$ is a spanning vector-valued measure, then there exists a unitary operator $U\colon \mathbb{C}^2 \to H$ such that $U\mathcal{E}(A) = \mathcal{E}'(A)$ for all $A \in \mathcal{A}_n$.*

Corollary 2.4. (a) $D_n(\Omega_n, \Omega_n) = 1$. (b) $A \mapsto D_n(A,B)$ *is a complex-valued measure for every* $B \in \mathcal{A}_n$. (c) *If $A_1, \ldots, A_k$ are sets in $\mathcal{A}_n$, then the $k \times k$ matrix $D_n(A_i, A_j)$, $i, j = 1, \ldots, k$ is positive semi-definite ("strong positivity").*

Proof. (a) follows from Theorem 2.1 and the definition of D_n while (b) follows from the definition of D_n. To verify (c), let $A_1, \ldots, A_k \in \mathcal{A}_n$ and let $a_1, \ldots, a_k \in \mathbb{C}$. Then by Theorem 2.3 we have that

$$\begin{aligned}\sum_{i,j=1}^{k} D_n(A_i, A_j) a_i \overline{a_j} &= \sum_{i,j=1}^{k} \langle \mathcal{E}(A_i), \mathcal{E}(A_j)\rangle a_i \overline{a_j} \\ &= \left\langle \sum_{i=1}^{k} a_i \mathcal{E}(A_i), \sum_{j=1}^{k} a_j \mathcal{E}(A_j) \right\rangle \geq 0\end{aligned}$$ □

It follows from Corollary 2.4(c) that $\mu_n(A) = D_n(A,A) \geq 0$ for all $A \in \mathcal{A}_n$, $\mu_n(\Omega_n) = 1$ and by inspection $\mu_n(\omega) = 1/2^n$ for all $\omega \in \Omega_n$. The next result is proved in [7, 12, 13].

Theorem 2.5. *The n-truncated q-measure μ_n satisfies the following conditions.* (a) *(grade-2 additivity) For mutually disjoint $A, B, C \in \mathcal{A}_n$ we have* $\mu_n(A \cup B \cup C) = \mu_n(A \cup B) + \mu_n(A \cup C) + \mu_n(B \cup C) - \mu_n(A) - \mu_n(B) - \mu_n(C)$ (b) *(regularity) If $\mu_n(A) = 0$, then $\mu_n(A \cup B) = \mu_n(B)$ whenever $A \cap B = \emptyset$. If $A \cap B = \emptyset$ and $\mu_n(A \cup B) = 0$, then $\mu_n(A) = \mu_n(B)$.*

It follows from Theorem 2.5(a) and induction that for $3 \leq m \leq n$ we have for $\{i_1, \ldots, i_m\} \subseteq \Omega_n$

$$\mu_n(\{i_1, \ldots, i_m\}) = \sum_{j<k=1}^{m} \mu_n(\{i_j, i_k\}) - (m-2) \sum_{j=1}^{m} \mu_n(i_j) \tag{2.4}$$

We have seen that

$$\mu_n(\{i,j\}) = \tfrac{1}{2^{n-1}} + 2D_{ij}^n$$

Applying Theorem 2.1 we conclude that

$$\mu_n(\{i,j\}) = \begin{cases} 1/2^{n-1} & \text{if } inj \\ 1/2^{n-2} & \text{if } icj \\ 0 & \text{if } idj \end{cases} \tag{2.5}$$

An event $A \in \mathcal{A}_n$ is *precluded* if $\mu_n(A) = 0$. The coevent (or anhomomorphic logic) interpretation of the path-integral [3, 8, 9, 13, 14, 16] confers a special importance on the precluded events. We shall show that precluded events are relatively rare. As an illustration, consider Examples 1 and 2. Besides the empty set $\emptyset$ there are no precluded events in $\mathcal{A}_1$ and the only precluded event in $\mathcal{A}_2$ is $\{0,2\}$.

If $m \leq n$ then we can consider Ω_m as a subset of Ω_n by padding on the right with zeros; and this in turn would let us consider subsets of Ω_m as subsets of Ω_n. If this is done, it follows from (2.4) and (2.5) that for $A \in \mathcal{A}_m$ we have that

$$\mu_m(A) = 2^{n-m}\mu_n(A) \tag{2.6}$$

Thus, if A is precluded in $\mathcal{A}_m$ then A is precluded in $\mathcal{A}_n$ for all $n \geq m$. However, this embedding of $\mathcal{A}_m$ into $\mathcal{A}_n$ is not unique, nor is it the most natural way to proceed when the elements of $\mathcal{A}_m$ and $\mathcal{A}_n$ are thought of as events. Rather one would regard Ω_m as a *quotient* of Ω_n, identifying an event in Ω_m with its lift to Ω_n. This is the the point of view adopted implicitly in the following section.

Lemma 2.6. *If $A \in \mathcal{A}_n$ has odd cardinality, then A is not precluded.*

Proof. Suppose $A = \{i_1, \ldots, i_m\} \in \mathcal{A}_n$ where m is odd. If $\mu_n(A) = 0$ then applying (2.4) gives

$$\sum_{j<k=1}^{m} \mu_n(\{i_j, i_k\}) = \frac{(m-2)m}{2^n} \tag{2.7}$$

where we are assuming that $m \geq 3$ because singleton sets are not precluded. Notice that $(m-2)m$ is odd. However, by (2.5) the left side of (2.7) has the form $r/2^n$ where r is even. This is a contradiction. Hence, $\mu_n(A) \neq 0$ so A is not precluded. □

Example 5. For $n = 3$ we have $\Omega_3 = \{0, 1, \ldots, 7\}$. Since $|\mathcal{A}_3| = 2^8$ is large, it is impractical to find by hand $\mu_3(A)$ for all $A \in \mathcal{A}_3$ so we shall just compute some of them. Of course, $\mu_3(\emptyset) = 0$ and $\mu(i) = 1/8$, $i = 0, 1, \ldots, 7$. By (2.5) we have that

$$\mu_3(\{i,j\}) = \begin{cases} 1/4 & \text{if } inj \\ 1/2 & \text{if } icj \\ 0 & \text{if } idj \end{cases}$$

Of the 28 doubleton sets the only precluded ones are: $\{0,2\}$, $\{0,4\}$, $\{0,6\}$, $\{1,5\}$, $\{3,5\}$, $\{5,7\}$. By Lemma 2.6 there are no precluded tripleton sets $A = \{i,j,k\}$. By (2.4) we have

$$\mu_3(A) = \mu_3(\{i,j\}) + \mu_3(\{i,k\}) + \mu_3(\{j,k\}) - \tfrac{3}{8}$$

The possibilities are: $inj, ink, jdk, \mu_3(A) = 1/8$; $inj, ink, jck, \mu_3(A) = 5/8$; $icj, ick, jck, \mu_3(A) = 9/8$ and the other possibilities coincide with one of these by symmetry. For a set of cardinality 4, $A = \{i,j,k,l\}$ and by (2.4) we have

$$\begin{aligned} \mu_3(A) = {} & \mu_3(\{i,j\}) + \mu_3(\{i,k\}) + \mu_3(\{i,l\}) + \mu_3(\{j,k\}) \\ & + \mu_3(\{j,l\}) + \mu_3(\{k,l\}) - 1 \end{aligned}$$

The possibilities are:
$inj, ink, inl, jdk, jdl, kcl, \mu_3(A) = 1/4$; $inj, ink, inl, jck, jcl, lck, \mu_3(A) = 3/4$; $inj, ink, idl, jdk, jnl, knl, \mu_3(A) = 0$; $inj, ink, idl, jck, jnl, knl, \mu_3(A) = 1/2$; $inj, ink, icl, jck, jnl, knl, \mu_3(A) = 1$; $idj, idk, idl, jck, jcl, kcl, \mu(A) = 1/2$, $idj, ick, icl, jdk, jdl, kcl, \mu(A) = 1/2$. The other possibilities coincide with one of these by symmetry. Of the 70 sets of cardinality 4 the only precluded ones are: $\{0,2,1,5\}$, $\{0,2,3,5\}$, $\{0,2,5,7\}$, $\{0,4,1,5\}$, $\{0,4,3,5\}$, $\{0,4,5,7\}$, $\{0,6,1,5\}$, $\{0,6,3,5\}$, $\{0,6,5,7\}$. There are no precluded events of cardinality > 4 in Ω_3.

Example 6. We compute some q-measures of events in $\Omega_4 = \{0, 1, \ldots, 15\}$. Some precluded doubleton sets are $\{0,2\}$, $\{0,4\}$, $\{2,10\}$, $\{4,10\}$. Moreover,

$$\mu_4(\{0,10\}) = \mu_4(\{2,4\}) = 1/4$$

It follows from (2.4) that $\{0,2,4,10\}$ is precluded. (In view of Theorem 2.5 (b), this also follows from the fact that $\{0,2,4,10\}$ is the disjoint union of the two precluded sets, $\{0,2\}$ and $\{4,10\}$.)

3 Cylinder Sets

For $\omega \in \Omega_n$ we identify the pair $(\omega, 0)$ with the string $\omega 0 \in \Omega_{n+1}$ obtained by adjoining 0 to the right of the string ω. Similarly we identify $(\omega, 1)$ with $\omega 1 \in \Omega_{n+1}$. For example, $(011, 0) = 0110$ and $(011, 1) = 0111$. We can also identify $\omega \times \{0, 1\}$ with the set $\{(\omega, 0), (\omega, 1)\} \in \mathcal{A}_{n+1}$. In a similar way, for $A \in \mathcal{A}_n$ we define $A \times \{0, 1\} \in \mathcal{A}_{n+1}$, by

$$A \times \{0, 1\} = \cup \{\omega \times \{0, 1\} : \omega \in A\}$$

Lemma 3.1. *If $A \in \mathcal{A}_n$, then $\mu_{n+1}(A \times \{0, 1\}) = \mu_n(A)$.*

Proof. For $\omega \in \Omega_n$, let $a(\omega) = i^{c_n(\omega)}$. By (2.2) we have

$$D_n(\omega, \omega') = \tfrac{1}{2^n}\, a(\omega)\overline{a(\omega')}\delta_{\alpha_n \alpha'_n}$$

Hence,

$$\begin{aligned}
&\mu_{n+1}(A \times \{0, 1\}) \\
&\quad = D_{n+1}(A \times \{0, 1\}, A \times \{0, 1\}) \\
&\quad = \sum \{D_{n+1}(\omega, \omega') \colon \omega, \omega' \in A \times \{0, 1\}\} \\
&\quad = \sum \{D_{n+1}(\omega 0, \omega' 0) \colon \omega, \omega' \in A\} \\
&\qquad + \sum \{D_{n+1}(\omega 1, \omega' 1) \colon \omega, \omega' \in A\} \\
&\quad = \frac{1}{2^{n+1}} \left[\sum \left\{ a(\omega)\overline{a(\omega')} \colon \omega, \omega' \in A, \alpha_n, \alpha'_n = 0 \text{ or } \alpha_n, \alpha'_n = 1 \right\} \right. \\
&\qquad + i \sum \left\{ a(\omega)\overline{a(\omega')} \colon \omega, \omega' \in A, \alpha_n = 1, \alpha'_n = 0 \right\} \\
&\qquad - i \sum \left\{ a(\omega)\overline{a(\omega')} \colon \omega, \omega' \in A, \alpha_n = 0, \alpha'_n = 1 \right\} \\
&\qquad + \sum \left\{ a(\omega)\overline{a(\omega')} \colon \omega, \omega' \in A, \alpha_n, \alpha'_n = 0 \text{ or } \alpha_n, \alpha'_n = 1 \right\} \\
&\qquad - i \sum \left\{ a(\omega)\overline{a(\omega')} \colon \omega, \omega' \in A, \alpha_n = 1, \alpha'_n = 0 \right\} \\
&\qquad \left. + i \sum \left\{ a(\omega)\overline{a(\omega')} \colon \omega, \omega' \in A, \alpha_n = 0, \alpha'_n = 1 \right\} \right] \\
&\quad = \frac{1}{2^n} \sum \left\{ a(\omega)\overline{a(\omega')} \colon \omega, \omega' \in A, \alpha_n, \alpha'_n = 0 \text{ or } \alpha_n, \alpha'_n = 1 \right\} \\
&\quad = \sum \{D_n(\omega, \omega') \colon \omega, \omega' \in A\} = \mu_n(A)
\end{aligned}$$

□

We use the notation $A^n = A \times A \cdots \times A$ (n factors).

Corollary 3.2. *If $A \in \mathcal{A}_n$, then $\mu_{n+m}(A \times \{0,1\}^m) = \mu_n(A)$.*

Corollary 3.3. *If $A \in \mathcal{A}_n$ is precluded, then $A \times \{0,1\}^m$ is also precluded.*

Remark Lemma 3.1 and its corollaries are valid for any finite unitary system in the sense of [5]. Indeed Corollary 3.3 can be seen as a special case of a much stronger assertion that holds for such systems: *If A is precluded and B is any subsequent event then the event $C = (A$ and $B)$ is also precluded.* Here, the condition that B be *subsequent to A*, means more precisely the following. By definition any event A is a set of histories or paths, and if these paths are singled out by a condition that concerns their behavior only for times $t < t_0$, we will say that A is "earlier than" t_0. Defining events later than t_0 analogously, we then say that B is subsequent to A if for some t_0, A is earlier than t_0 and B is later. The event (A and B) is of course simply the intersection $A \cap B$ expressed as a logical conjunction. We can also write it as (A and-then B) in order to emphasize that B is meant to be subsequent to A. The preservation of preclusion by 'and-then' can be viewed as a kind of causality condition. When this condition is fulfilled, one can correlate to any event B later than t_0 and earlier than t_1 a linear operator from the Hilbert space associated with times $t < t_0$ to that associated with times $t < t_1$. In a situation like that of Cor. 3.3, the earlier (resp. later) Hilbert space would be that associated to $\mathcal{A}_n$ (resp. $\mathcal{A}_{n+m}$).

Example 7. In Example 2 we saw that $\{0,2\} \in \mathcal{A}_2$ is precluded. Applying Corollary 3.3 shows that $\{0,4,1,5\} \in \mathcal{A}_3$ is also precluded. Applying Corollary 3.3 again shows that $\{0,8,2,10,1,9,3,11\} \in \mathcal{A}_4$ is precluded.

Using our previously established notation we can write $\Omega = \{0\} \times \{0,1\} \times \cdots$ or $\Omega = \Omega_n \times \{0,1\} \times \{0,1\} \times \cdots$. A subset $A \subseteq \Omega$ is a *cylinder set* if there exists a $B \in \mathcal{A}_n$ for some $n \in \mathbb{N}$ such that $A = B \times \{0,1\} \times \{0,1\} \times \cdots$. Thus, the first $n+1$ bits for strings in A are restricted and the further bits are not. For $\omega \in \Omega_n$, we call $\text{cyl}(\omega) = \{\omega\} \times \{0,1\} \times \{0,1\} \times \cdots$ an *elementary cylinder set.*[2] If $\omega = \alpha_0\alpha_1 \cdots \alpha_m \in \Omega_m$ and $\omega' = \alpha_0\alpha_1 \cdots \alpha_m\alpha_{m+1} \cdots \alpha_n \in \Omega_n$, $m \le n$ we say that ω' is an *extension* of ω. We have that $\text{cyl}(\omega') \subseteq \text{cyl}(\omega)$ if and only if ω' is an extension of ω and $\text{cyl}(\omega') \cap \text{cyl}(\omega) = \emptyset$ if and only if neither ω or ω' is an extension of the other. (Thus any two elementary cylinder sets

[2]In references [1, 2, 15] the term cylinder-set is reserved for what are here called *elementary* cylinder sets.

are either disjoint or nested.) Moreover, any cylinder set is a finite disjoint union of elementary cylinder sets.

We denote the collection of all cylinder sets by $\mathcal{C}(\Omega) = \mathcal{C}$. If $A \in \mathcal{C}$ then its complement A' is clearly in $\mathcal{C}$. Similarly, $\mathcal{C}$ is closed under finite unions and finite intersections so $\mathcal{C}$ is an algebra of subsets of Ω. Of course, there are subsets of Ω that are not in $\mathcal{C}$; for example, $\{\omega\} \notin \mathcal{C}$ for $\omega \in \Omega$. For $A \in \mathcal{C}$ if $A = B \times \{0,1\} \times \{0,1\} \times \cdots$ with $B \in \mathcal{A}_n$ we define $\mu(A) = \mu_n(B)$. To show that $\mu\colon \mathcal{C} \to \mathbb{R}^+$ is well-defined, suppose $A = B_1 \times \{0,1\} \times \{0,1\} \times \cdots$ with $B_1 \in \mathcal{A}_m$. If $m = n$, then $B = B_1$ and we're finished. Otherwise, we can assume that $m < n$. It follows that $B = B_1 \times \{0,1\}^{n-m}$. Hence, $\mu_n(B) = \mu_m(B_1)$ by Corollary 3.2 so μ is well-defined. It is clear that $\mu\colon \mathcal{C} \to \mathbb{R}^+$ satisfies Conditions (a) and (b) of Theorem 2.5 so we can consider μ as a q-measure on $\mathcal{C}$.

As before, we say that $A \in \mathcal{C}$ is *precluded* if $\mu(A) = 0$. We also say that $B \in \mathcal{C}$ is *stymied* if $B \subseteq A$ for some precluded $A \in \mathcal{C}$. Of course a precluded set is stymied but there are many stymied sets that are not precluded. For instance, by Example 2, cyl(000) and cyl(101) are not precluded but are stymied. It is clear that $\mu(\Omega) = 1$ and Ω is not stymied. Surprisingly, it is shown in [5] that Ω is the only set in $\mathcal{C}$ that is not stymied.

Let $A_1 \supseteq A_2 \supseteq \cdots$ be a decreasing sequence in $\mathcal{C}$ with $A = \cap A_i \in \mathcal{C}$. (In general, A need not be in $\mathcal{C}$.) We shall show in the proof of Theorem 4.1 that Ω is compact in the product topology and that every element of $\mathcal{C}$ is compact. Letting $B_i = A_i \smallsetminus A$, since $B_i \in \mathcal{C}$ we conclude that B_i is compact, $i = 1, 2, \ldots$, and that $B_1 \supseteq B_2 \supseteq \cdots$ with $\cap B_i = \emptyset$. It follows that $B_m = \emptyset$ for some $m \in \mathbb{N}$. Hence, $A_m = A$ so $A_i = A$ for $i \geq m$. We conclude that

$$\lim \mu(A_i) = \mu\left(\cap A_i\right) \tag{3.1}$$

Now let $A_1 \subseteq A_2 \subseteq \cdots$ be an increasing sequence in $\mathcal{C}$ with $\cup A_i \in \mathcal{C}$. By taking complements of our previous work we have

$$\lim_{i \to \infty} \mu(A_i) = \mu\left(\cup A_i\right) \tag{3.2}$$

Since μ satisfies (3.1) and (3.2) we say that μ is *continuous* on $\mathcal{C}$.

Let $\mathcal{A}$ be the σ-algebra generated by $\mathcal{C}$. If μ were a finitely additive probability measure on $\mathcal{C}$ satisfying (3.1) or (3.2) , then by the Kolmogorov extension theorem, μ would have a unique extension to a (countably additive) probability measure on $\mathcal{A}$. The next example shows that this extension theorem does not hold for q-measures; that is, μ does not have an extension to a continuous q-measure on $\mathcal{A}$.

Example 8. Let

$$B_1 = \{0010, 0100, 0110\} = \{2, 4, 6\} \in \mathcal{A}_3$$

As in Example 5 $\mu_3(B_1) = 9/8$. Letting $B_2 = \{010, 100, 110\}$ we have that

$$B_1 \times B_2 = \{0010, 0100, 0110\} \times \{010, 100, 110\} \in \mathcal{A}_6$$

A simple calculation shows that $\mu_6(B_1 \times B_2) = (9/8)^2$ and continuing, $\mu_9(B_1 \times B_2 \times B_2) = (9/8)^3$. Defining $A_i \in \mathcal{C}$ by $A_1 = B_1 \times \{0,1\} \times \{0,1\} \times \cdots$, $A_2 = B_1 \times B_2 \times \{0,1\} \times \{0,1\} \times \cdots$, $A_3 = B_1 \times B_2 \times B_2 \times \{0,1\} \times \{0,1\} \times \cdots$ we have that $A_1 \supseteq A_2 \supseteq \cdots$. However, $\mu(A_i) = (9/8)^i$ so $\lim_{i\to\infty} \mu(A_i) = \infty$. Hence, if μ had an extension to $\mathcal{A}$, then (3.1) would fail.

Another way to show that μ does not extend to $\mathcal{A}$ is given in [5]. We define the *total variation* $|\mu|$ of μ by

$$|\mu|(A) = \left[\sup_{\pi(A)} \sum_i \mu(A_i)^{1/2}\right]^2$$

for all $A \in \mathcal{C}$ where the supremum is over all finite partitions $\pi(A) = \{A_1, \ldots, A_n\}$ of A with $A_i \in \mathcal{C}$. We say that μ is of *bounded variation* if $|\mu(A)| < \infty$ for all $A \in \mathcal{C}$. It is shown in [5] that if μ has an extension to a continuous q-measure on $\mathcal{A}$, then μ must be of bounded variation. It is proved in [5] that for any finite unitary system, μ is not of bounded variation. Although this proof is difficult for our particular case it is simple.

Example 9. We show that μ is not of bounded variation. For $0, 1, \ldots, 2^n - 1 \in \Omega_n$ we have the partition of Ω

$$\Omega = \bigcup_{i=1}^{2^n-1} \mathrm{cyl}(i)$$

and

$$\sum_{i=0}^{2^n-1} \mu\left[\mathrm{cyl}(i)\right]^{1/2} = \sqrt{2^n}$$

Hence, $|\mu|(\Omega) \geq 2^n$ for all $n \in \mathbb{N}$ so $|\mu|(\Omega) = \infty$. A similar argument shows that $|\mu|(A) = \infty$ for all $A \in \mathcal{C}$, $A \neq \emptyset$.

Although we cannot extend μ to a continuous q-measure on $\mathcal{A}$, perhaps we can extend μ to physically interesting sets in $\mathcal{A} \smallsetminus \mathcal{C}$. We now discuss a possible way to accomplish this.[3] For $\omega = \alpha_0\alpha_1 \cdots \in \Omega$ and $A \subseteq \Omega$ we write $\omega(n)A$ if there is an $\omega' = \beta_0\beta_1 \cdots \in A$ such that $\beta_i = \alpha_i$, $i = 0, 1, \ldots, n$. We then define

$$A^{(n)} = \{\omega \in \Omega \colon \omega(n)A\}$$

Notice that $A^{(n)} \in \mathcal{C}$, $A^{(0)} \supseteq A^{(1)} \supseteq A^{(2)} \supseteq \cdots$, and $A \subseteq \cap A^{(n)}$. We think of $A^{(n)}$ as a particular sort of time-n cylindrical approximation to A. We say that $A \subseteq \Omega$ is a *lower set* if $A = \cap A^{(n)}$; and we call A *beneficial* if $\lim \mu\left(A^{(n)}\right)$ exists and is finite. We denote the collection of lower sets by $\mathcal{L}$, the collection of beneficial sets by $\mathcal{B}$ and write $\mathcal{B}_{\mathcal{L}} = \mathcal{B} \cap \mathcal{L}$. If $A \in \mathcal{B}$, we define $\widehat{\mu}(A) = \lim \mu\left(A^{(n)}\right)$.

The next section considers algebraic structures but for now we mention that Example 9 to follow shows that $\mathcal{L}$ is not closed under $'$ so is not an algebra. Since $\mathcal{A}$ is closed under countable intersections, $\mathcal{L} \subseteq \mathcal{A}$. If $A \in \mathcal{C}$, then $A = A^{(n)} = A^{(n+1)} = \cdots$ for some $n \in \mathbb{N}$. Hence, $A = \cap A^{(n)}$ and $\mu(A) = \lim \mu\left(A^{(n)}\right) = \widehat{\mu}(A)$. Thus, $\mathcal{C} \subseteq \mathcal{B}_{\mathcal{L}}$ and the definition of $\widehat{\mu}$ on $\mathcal{B}_{\mathcal{L}}$ reduces to the usual definition of μ on $\mathcal{C}$. The following result shows that $\{\omega\} \in \mathcal{B}_{\mathcal{L}}$ and $\widehat{\mu}(\{\omega\}) = 0$ for all $\omega \in \Omega$. We conclude that $\mathcal{B}_{\mathcal{L}}$ properly contains $\mathcal{C}$.

Lemma 3.4. *If $A \subseteq \Omega$ with $|A| < \infty$, then $A \in \mathcal{B}_{\mathcal{L}}$ and $\widehat{\mu}(A) = 0$.*

Proof. Suppose that $\omega = \alpha_0\alpha_1 \cdots \notin A$. Then there exists an $n \in \mathbb{N}$ such that $\alpha_0\alpha_1 \cdots \alpha_n$ is different from the first n bits of all $\omega' \in A$. But then $\omega \notin A^{(n)}$ so $\omega \notin \cap A^{(n)}$. Hence, $A = \cap A^{(n)}$. If $|A| = m$, then $A^{(n)} = B_n \times \{0,1\} \times \{0,1\} \times \cdots$, $B_n \in \mathcal{A}_n$ with $|B_n| \leq m$, $n = 0, 1, 2, \ldots$. Hence

$$\mu(A^{(n)}) = \mu_n(B_n) = D_n(B_n, B_n) = \sum \{D_n(\omega, \omega') \colon \omega, \omega' \in B_n\} \leq \tfrac{m^2}{2^n}$$

Hence, $\lim \mu\left(A^{(n)}\right) = 0$. We conclude that $A \in \mathcal{B}_{\mathcal{L}}$ and $\widehat{\mu}(A) = 0$. □

Example 10. Let $A \subseteq \Omega$ with $|A| < \infty$, $A \neq \emptyset$. We then have that $A'^{(n)} = \Omega$, $n = 0, 1, 2, \ldots$. Hence, $A' \neq \cap A'^{(n)} = \Omega$ so $A' \notin \mathcal{L}$. This shows that $\mathcal{L}$ is not an algebra. This also shows that $\mathcal{B}_{\mathcal{L}}$ is not an algebra.

[3] Some of the ideas expressed here in embryo are developed further in [15].

Example 11. Define the set

$$A = \{\omega \in \Omega \colon \omega \text{ has at most one } 1\}$$

We have that

$$A^{(n)} = \{00\cdots 0, 010\cdots 0, 0010\cdots 0, \cdots, 00\cdots 01\}$$

It is clear that $A = \cap A^{(n)}$. Also, we have

$$\mu\left(A^{(n)}\right) = \tfrac{1}{2^n}\left[n+1-2(n+1)+2\binom{n-1}{2}\right] = \tfrac{1}{2^n}(n^2-4n+5)$$

Hence, $\lim \mu\left(A^{(n)}\right) = 0$ so $A \in \mathcal{B}_{\mathcal{L}}$ and $\widehat{\mu}(A) = 0$.

4 Quadratic Algebras

This section discusses algebraic structures for the collections $\mathcal{L}$, $\mathcal{B}$ and $\mathcal{B}_{\mathcal{L}}$. A collection Q of subsets of a set S is a *quadratic algebra* if $\emptyset, S \in Q$ and if $A, B, C \in Q$ are mutually disjoint and $A \cup B, A \cup C, B \cup C \in Q$, then $A \cup B \cup C \in Q$. If Q is a quadratic algebra, a *q-measure* on Q is a map $\nu \colon Q \to \mathbb{R}^+$ such that if $A, B, C \in Q$ are mutually disjoint and $A \cup B, A \cup C, B \cup C \in Q$, then

$$\nu(A \cup B \cup C) = \nu(A \cup B) + \nu(A \cup C) + \nu(B \cup C) - \nu(A) - \nu(B) - \nu(C)$$

Example 12. Let $S = \{d_1, d_2, d_3, u_1, u_2, u_3, s_1, s_2, s_3\}$ and define $Q \subseteq 2^S$ by $\emptyset, S \in Q$ and $A \in Q$ if and only if each of the three types of elements have different cardinalities in A, $A \neq \emptyset, S$. For instance,

$$\{u_1, d_1, d_2\}, \{u_1, d_1, d_2, s_1, s_2, s_3\} \in Q$$

and these are the only kinds of sets in Q besides $\emptyset, S$. Although Q is closed under complementation, it is not closed under disjoint unions so Q is not an algebra. For instance $\{u_1, d_1, d_2\}, \{u_2, s_1, s_2\} \in Q$ but

$$\{u_1, u_2, d_1, d_2, s_1, s_2\} \notin Q$$

To show that Q is a quadratic algebra, suppose $A, B, C \in Q$ are mutually disjoint and $A \cup B, A \cup C, B \cup C \in Q$. If one or more of A, B, C are empty then clearly, $A \cup B \cup C \in Q$ so suppose $A, B, C \neq \emptyset$. Since $|A| = |B| = |C| = 3$, we have $A \cup B \cup C = S \in Q$. An example of a q-measure on Q is $\nu(\emptyset) = 0$, $\nu(S) = 1$, $\nu(A) = 1/6$ if $|A| = 3$ and $\nu(A) = 1/2$ if $|A| = 6$. If A, B, C are mutually disjoint nonempty sets in Q, then $\nu(A \cup B \cup C) = \nu(\Omega) = 1$ and

$$\nu(A \cup B) + \nu(A \cup C) + \nu(B \cup C) - \nu(A) - \nu(B) - \nu(C) = \tfrac{3}{2} - \tfrac{1}{2} = 1$$

Notice that ν is not additive because

$$\nu\left(\{u_1, d_1, d_2\}\right) + \nu\left(\{u_2, u_3, s_1\}\right) = \tfrac{1}{3} \neq \tfrac{1}{2} = \nu\left(\{u_1, u_2, u_3, d_1, d_2, s_1\}\right)$$

Example 13. Let $S = \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ where n is odd. Let

$$Q = \{A \subseteq S \colon |\{x_i \colon x_i \in A\}| = 0 \text{ or odd}\}$$

Notice that Q is not closed under complementation, unions (even disjoint unions) or intersections. To show that Q is a quadratic algebra, suppose $A, B, C \in Q$, are mutually disjoint and $A \cup B, A \cup C, B \cup C \in Q$. Since $A \cup C, B \cup C \in Q$ at most one of A, B, C has an odd number of x_is and the other contain no x_is. Hence, $A \cup B \cup C \in Q$. An example of a nonadditive q-measure on Q is $\nu(A) = |A|^2$. In fact, $\nu(A) = |A|^2$ is a q-measure on any finite quadratic algebra.

Theorem 4.1. *$\mathcal{L}$ and $\mathcal{B}_{\mathcal{L}}$ are quadratic algebras and $\widehat{\mu}$ is a q-measure on $\mathcal{B}_{\mathcal{L}}$ that extends μ on $\mathcal{C}$.*

Proof. Placing the discrete topology on $\{0, 1\}$, since $\{0, 1\}$ is compact, by Tychonov's theorem $\Omega = \{0, 1\} \times \{0, 1\} \times \cdots$ is compact in the product topology. Any cylinder set is closed (and open) and hence is compact. Let $A, B \in \mathcal{L}$ with $A \cap B = \emptyset$. Since $A = \cap A^{(n)}$, $B = \cap B^{(n)}$ we have that

$$\cap\left(A^{(n)} \cap B^{(n)}\right) = \left(\cap A^{(n)}\right) \cap \left(\cap B^{(n)}\right) = A \cap B = \emptyset$$

Since $A^{(n)} \cap B^{(n)}$ is a decreasing sequence of compact sets with empty intersection, there exists an $n \in \mathbb{N}$ such that $A^{(m)} \cap B^{(m)} = \emptyset$ for $m \geq n$. Now let $A, B, C \in \mathcal{L}$ be mutually disjoint with $A \cup B, A \cup C, B \cup C \in \mathcal{L}$. By our previous work there exists an $n \in \mathbb{N}$ such that $A^{(m)}, B^{(m)}, C^{(m)}$ are mutually disjoint for $m \geq n$. By the distributive law we have

$$\begin{aligned} A \cup B \cup C &= \left(\cap A^{(m)}\right) \cup \left(\cap B^{(m)}\right) \cup \left(\cap C^{(m)}\right) \\ &= \cap\left(A^{(m)} \cup B^{(m)} \cup C^{(m)}\right) = \cap\left[(A \cup B \cup C)^{(m)}\right] \end{aligned}$$

Hence, $\mathcal{L}$ is a quadratic algebra. If $A, B, C, A \cup B, A \cup C, B \cup C \in \mathcal{B}_{\mathcal{L}}$ with A, B, C disjoint, since $A^{(m)}, B^{(m)}, C^{(m)}$ are eventually disjoint we conclude that

$$\begin{aligned}\lim \mu \left[(A \cup B \cup C)^{(m)}\right] &= \lim \mu \left[A^{(m)} \cup B^{(m)} \cup C^{(m)}\right] \\ &= \lim \mu \left(A^{(m)} \cup B^{(m)}\right) + \lim \mu \left(A^{(m)} \cup C^{(m)}\right) + \lim \left(B^{(m)} \cup C^{(m)}\right) \\ &\quad - \lim \mu(A^{(m)}) - \lim \mu(B^{(m)}) - \lim \mu(C^{(m)}) \\ &= \widehat{\mu}(A \cup B) + \widehat{\mu}(A \cup C) + \widehat{\mu}(B \cup C) - \widehat{\mu}(A) - \widehat{\mu}(B) - \widehat{\mu}(C)\end{aligned}$$

Hence, $A \cup B \cup C \in \mathcal{B}_{\mathcal{L}}$ so $\mathcal{B}_{\mathcal{L}}$ is a quadratic algebra. Also,

$$\widehat{\mu}(A \cup B \cup C) = \widehat{\mu}(A \cup B) + \widehat{\mu}(A \cup C) + \widehat{\mu}(B \cup C) - \widehat{\mu}(A) - \widehat{\mu}(B) - \widehat{\mu}(C)$$

so $\widehat{\mu}$ is a q-measure on $\mathcal{B}_{\mathcal{L}}$ that extends μ on $\mathcal{C}$. □

We say that $A \subseteq \Omega$ is an *upper set* if $A = \cup A'^{(n)\prime}$ and denote the collection of upper sets by $\mathcal{U}$. Since $A'^{(n)}$ is a decreasing sequence of cylinder sets, we conclude if $A \in \mathcal{U}$ then A is the union of an increasing sequence of cylinder sets $A'^{(n)\prime}$. For example, if $|A| < \infty$ we have shown that $A \in \mathcal{L}$ so that $A = \cap A^{(n)}$. Hence,

$$A' = \cup A^{(n)\prime} = \cup (A')'^{(n)\prime}$$

It follows that $A' \in \mathcal{U}$ so $\mathcal{U}$ properly contains $\mathcal{C}$. Moreover, $A' \notin \mathcal{L}$ so $\mathcal{U} \notin \mathcal{L}$.

Lemma 4.2. *Suppose $B \subseteq \Omega$ and there exists a decreasing sequence $C_i \in \mathcal{C}$ and an increasing sequence $D_i \in \mathcal{C}$ such that*

$$B = \cap C_i = \cup D_i$$

Then $B \in \mathcal{C}$.

Proof. We have that $D_i \subseteq B \subseteq C_i$, $C_i \smallsetminus D_i \in \mathcal{C}$ and

$$\cap (C_i \smallsetminus D_i) = \cap (C_i \cap D_i') = (\cap C_i) \cap (\cap D_i') = B \cap B' = \emptyset$$

Since $C_i \smallsetminus D_i$ is compact in the product topology, there exists a $j \in \mathbb{N}$ such that $C_j \smallsetminus D_j = \emptyset$. Therefore, $D_j = C_j$. Since $D_j \subseteq B \subseteq C_j$, we have $B = D_j = C_j$ so that $B \in \mathcal{C}$. □

Corollary 4.3. (a) $\mathcal{L} \cap \mathcal{U} = \mathcal{C}$. (b) *If $A, A' \in \mathcal{L}$, then $A \in \mathcal{C}$.*

Proof. (a) This follows directly from Lemma 4.2. (b) Since $A \in \mathcal{L}$, we have that $A = \cap A^{(n)}$ where $A^{(n)} \in \mathcal{C}$ is decreasing. Since $A' \in \mathcal{L}$ we have that $A' = \cap A'^{(n)}$. Hence, $A = \cup A'^{(n)\prime}$ where $A'^{(n)\prime} \in \mathcal{C}$ is increasing. By Lemma 4.2, $A \in \mathcal{C}$. □

Theorem 4.4. (a) *If $A \in \mathcal{L}$, then $A' \in \mathcal{U}$.* (b) *$\mathcal{U}$ is a quadratic algebra.*

Proof. (a) If $A \in \mathcal{L}$, then $A = \cap A^{(n)}$ so that $A' = \cup (A')'^{(n)\prime}$. Hence, $A' \in \mathcal{U}$. (b) Clearly, $\emptyset, \Omega \in \mathcal{U}$. Suppose $A, B, C \in \mathcal{U}$ are mutually disjoint. Since

$$(A \cup B \cup C)'^{(n)} = (A' \cap B' \cap C')^{(n)} \subseteq (A')^{(n)} \cap (B')^{(n)} \cap (C')^{(n)}$$

we have that

$$A'^{(n)\prime} \cup B'^{(n)\prime} \cup C'^{(n)\prime} \subseteq (A \cup B \cup C)'^{(n)\prime}$$

Hence,

$$\begin{aligned} A \cup B \cup C &= \left(\cup A'^{(n)\prime}\right) \cup \left(\cup B'^{(n)\prime}\right) \cup \left(\cup C'^{(n)\prime}\right) \\ &= \cup \left(A'^{(n)\prime} \cup B'^{(n)\prime} \cup C'^{(n)\prime}\right) \subseteq \cup (A \cup B \cup C)'^{(n)\prime} \end{aligned}$$

But $(A \cup B \cup C)'^{(n)\prime} \subseteq A \cup B \cup C$ so that

$$A \cup B \cup C = \cup (A \cup B \cup C)'^{(n)\prime}$$

Therefore, $A \cup B \cup C \in \mathcal{U}$ so $\mathcal{U}$ is a q-algebra. □

Letting

$$\mathcal{B}_{\mathcal{U}} = \left\{A \in \mathcal{U} \colon \lim \mu_n(A'^{(n)\prime}) \text{ exists}\right\}$$

we see that $\mathcal{B}_{\mathcal{U}}$ is the "upper" counterpart of $\mathcal{B}_{\mathcal{L}}$. As before, if $A \in \mathcal{B}_{\mathcal{U}}$ we define $\widehat{\mu}(A) = \lim \mu_n(A'^{(n)\prime})$. Unfortunately, we have not been able to show that $\mathcal{B}_{\mathcal{U}}$ is a quadratic algebra. However, we shall show that $\{\gamma\}' \in \mathcal{B}_{\mathcal{U}}$ for $\gamma \in \Omega$. We first need the following lemma.

Lemma 4.5. *For $n \in \mathbb{N}$, $j = 0, 1, \ldots, 2^n - 1$, the function $c_n(j)$ satisfies*

$$c_{n+1}(2^{n+1} - 1 - j) = c_n(j) + 1$$

Proof. Let $j \in \Omega_n = \{0, 1, \ldots, 2^n - 1\}$ and for $a \in \{0, 1\}$, let $a' = a + 1 \pmod 2$. If j has binary representation $j = a_0 a_1 \cdots a_n$, $a_0 = 0$, $a_k \in \{0, 1\}$, $k = 1, \ldots, n$, since

$$a_0 a_1 \cdots a_n + a_0' a_1' \cdots a_n' = 2^{n+1} - 1$$

we have that

$$(2^{n+1}-1)-j = 0a'_0a'_1\cdots a'_n \in \Omega_{n+1}$$

Suppose that $c_n(j)=k$ so $a_0a_1\cdots a_n$ has k position switches. These position switches are in one-to-one correspondence with the position switches in $a'_0a'_1\cdots a'_n$. Since $a'_0=1$, $0a'_0a'_1\cdots a'_n$ has one more position switch so

$$c_{n+1}(2^{n+1}-j-1)=k+1 \qquad \square$$

Example 14. Since $c_1=(0,1)$, it follows immediately from Lemma 4.5 that $c_2=(0,1,2,1)$, $c_3=(0,1,2,1,2,3,2,1)$ and

$$c_4=(0,1,2,1,2,3,2,1,2,3,4,3,2,3,2,1)$$

To show that $\{\gamma\}'\in\mathcal{U}$, for simplicity let $\gamma=000\cdots$ and let $B=\{\gamma\}'$.

Theorem 4.6. *The set* $B\in\mathcal{B}_\mathcal{U}$ *and* $\widehat{\mu}(B)=1$.

Proof. For ease of notation, let $B_n=B'^{(n)\prime}=\{\gamma\}^{(n)\prime}$. Then $B_1=\{00\}'\times\{0,1\}\times\cdots$, $B_2=\{000\}\times\{0,1\}\times\cdots$, $\cdots$ and $B=\cup B_n\in\mathcal{U}$. We must show that $\lim\mu_n(B_n)=1$. From the definition of B_n we have that

$$\begin{aligned}\mu_n(B_n)&=\sum\left\{D^n_{\gamma,\gamma'}\colon\gamma,\gamma'\neq 0\right\}=1-\sum\left\{D^n_{\gamma,\gamma'}\colon\gamma\text{ or }\gamma'=0\right\}\\&=1-\frac{1}{2^n}-\frac{1}{2^{n-1}}\sum_{j=2}^{2^n-2}\left\{i^{c_n(j)}\colon j\text{ even}\right\}\\&=1-\frac{1}{2^n}-\frac{1}{2^{n-1}}\sum_{j=1}^{2^{n-1}-1}i^{c_n(2j)}=1+\frac{1}{2^n}-\frac{1}{2^{n-1}}\sum_{j=0}^{2^{n-1}-1}i^{c_n(2j)}\end{aligned}\tag{4.1}$$

Let $u_n(j)$ be the number of j-values of c_n. For example $u_3(0)=1$, $u_3(1)=3$, $u_3(2)=3$, $u_3(3)=1$. It follows from Lemma 4.5 that

$$u_{n+1}(j)=u_n(j)+u_n(j-1),\quad j=1,2,\ldots,n+1\tag{4.2}$$

Letting

$$v_n(j)=\sum\{u_n(k)\colon k=j\pmod 4\}$$

for $j=0,1,2,3$ we have that

$$v_n(j)=u_n(j)+u_n(j+4)+u_n(j+8)+\cdots+u_n\left(4\left\lfloor\frac{n-j}{4}\right\rfloor+j\right)$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Applying (4.2) we conclude that v_n satisfies the recurrence relations

$$v_{n+1}(j) = v_n(j) + v_n(j-1) \tag{4.3}$$

for $j = 1, 2, 3, 4$ where $v_n(-1) = v_n(3)$. Also, v_n satisfies the initial conditions $v_1(0) = v_1(1) = 1$, $v_1(2) = v_1(3) = 0$.

We now prove by mathematical induction on n that

$$v_n(j) = 2^{n-2} + 2^{\frac{n}{2}-1}\cos(n-2j)\pi/4 \tag{4.4}$$

By the initial conditions, (4.4) holds for $n = 1$, $j = 0, 1, 2, 3$. Suppose (4.4) holds for n and $j = 0, 1, 2, 3$. We then have by (4.3) that

$$\begin{aligned} v_{n+1}(j) &= v_n(j) + v_n(j-1) \\ &= 2^{n-1} + 2^{\frac{n}{2}-1}\left[\cos(n-2j)\pi/4 + \cos\left(n-2(j-1)\right)\pi/4\right] \\ &= 2^{n-1} + 2^{\frac{n}{2}-1}\left[\cos(n-2j)\pi/4 - \sin(n-2j)\pi/4\right] \\ &= 2^{n-1} + 2^{\frac{n}{2}-1}2^{1/2}\cos\left[(n-2j)\pi/4 + \pi/4\right] \\ &= 2^{(n+1)-2} - 2^{\frac{n+1}{2}-1}\cos\left[\left((n+1)-2j\right)\pi/4\right] \end{aligned}$$

This proves (4.4) by induction.

Applying (4.1) we have that

$$\mu_n(B_n) = 1 + \tfrac{1}{2^n} - \tfrac{1}{2^{n-1}}\left[v_n(0) - v_n(2)\right] \tag{4.5}$$

By (4.4) we have

$$v_n(0) = 2^{n-1} + 2^{\frac{n}{2}-1}\cos n\pi/4$$

and

$$v_n(2) = 2^{n-2} + 2^{\frac{n}{2}-1}\cos(n-4)\pi/4 = 2^{n-2} - 2^{\frac{n}{2}-1}\cos n\pi/4$$

Hence, (4.4) becomes

$$\mu_n(B_n) = 1 + \frac{1}{2^n} - \frac{2^{n/2}}{2^{n-1}}\cos n\pi/4 = 1 + \frac{1}{2^n} - \frac{1}{2^{n/2-1}}\cos n\pi/4$$

We conclude that $\lim \mu_n(B_n) = 1$. □

Notice that $C = \{0111\cdots\}'$ is the event that the particle ever returns to the site 0. Analogous to Theorem 4.6 we have that $C \in \mathcal{B}_{\mathcal{U}}$ and $\mu(C) = 1$. Thus, C is a physically significant event in $\mathcal{B}_{\mathcal{U}} \smallsetminus \mathcal{C}$ and the q-probability of return in unity.

We say that $A, B \subseteq \Omega$ are *strongly disjoint* if there is an $n \in \mathbb{N}$ such that $A^{(n)} \cap B^{(n)} = \emptyset$. It is clear that we then have that $A^{(m)} \cap B^{(m)} = \emptyset$ for $m \geq n$. Since $A \subseteq A^{(n)}$, $B \subseteq B^{(n)}$, if A and B are strongly disjoint, then $A \cap B = \emptyset$. However, the converse does not hold.

Example 15. Define $A \subseteq \Omega$ by

$$A = \{\omega \in \Omega \colon \omega \text{ has finitely many 1s}\}$$

Then $A \cap A' = \emptyset$ but $A^{(n)} = A'^{(n)} = \Omega$ for all $n \in \mathbb{N}$. Hence, $A^{(n)} \cap A'^{(n)} \neq \emptyset$ so A, A' are disjoint but not strongly disjoint.

A collection of subsets $Q \subseteq 2^{\Omega}$ is a *weak quadratic algebra* if $\emptyset, \Omega \in Q$ and if $A, B, C \in Q$ are strongly disjoint and $A \cup B, A \cup C, B \cup C \in Q$, then $A \cup B \cup C \in Q$. If Q is a weak quadratic algebra, a *q-measure* on Q is a map $\nu \colon Q \to \mathbb{R}^+$ such that if $A, B, C \in Q$ are strongly disjoint and $A \cup B, A \cup C, B \cup C \in Q$, then

$$\nu(A \cup B \cup C) = \nu(A \cup B) + \nu(A \cup C) + \nu(B \cup C) - \nu(A) - \nu(B) - \nu(C)$$

The proof of the next theorem is similar to that of Theorem 4.1.

Theorem 4.7. *$\mathcal{B}$ and $\mathcal{B}_{\mathcal{U}}$ are weak quadratic algebras and $\widehat{\mu}$ is a q-measure on $\mathcal{B}$ that extends μ to $\mathcal{B}$.*

Let A be the set in Example 15. We have that $A, A' \in \mathcal{B}$ and $\widehat{\mu}(A) = \widehat{\mu}(A') = 1$. Since $A, A' \notin \mathcal{L}$, we see that $\mathcal{B}$ properly contains $\mathcal{B}_{\mathcal{L}}$. We also have that the set $\mathcal{B}$ of Theorem 4.6 is in $\mathcal{B}$ but not in $\mathcal{L}$.

5 "Expectations"

This section explores the mathematical analogy between functions on Ω and random variables in classical probability theory, using a notion of "expectation" introduced in [6, 10]. When applied to the characteristic function χ_A of an event $A \in \mathcal{A}$, this expectation reproduces the quantum measure $\mu(A)$ of A, which classically would be the probability that the event A occur. One knows that such an interpretation is not viable quantum mechanically when interference is present; and one must seek elsewhere for the physical meaning of μ [16]. We hope that the formal relationships we expose here can be helpful in this quest, or in making further contact with the more traditional quantum formalism.

The following paragraphs consider expectations in terms of a q-integral [6, 10]. For a positive random variable $f\colon \Omega_n \to \mathbb{R}^+$ we define

$$\begin{aligned}\int f d\mu_n &= \sum_{i,j=0}^{2^n-1} \min\left[f(\omega_i), f(\omega_j)\right] D_n(\omega_i, \omega_j)\\ &= \sum_{i,j=0}^{2^n-1} \min\left[f(\omega_i), f(\omega_j)\right] D_{ij}^n \end{aligned}\tag{5.1}$$

An arbitrary random variable $f\colon \Omega_n \to \mathbb{R}$ has a unique representation $f = f^+ - f^-$ where $f^+, f^- \geq 0$ and $f^+ f^- = 0$ and we define

$$\int f d\mu_n = \int f^+ d\mu_n - \int f^- d\mu_n$$

This q-integral has the following properties. If $f \geq 0$, then $\int f d\mu_n \geq 0$, $\int \alpha f d\mu_n = \alpha \int f d\mu_n$ for all $\alpha \in \mathbb{R}$, $\int \chi_A d\mu_n = \mu_n(A)$ for all $A \in \mathcal{A}_n$ where χ_A is the characteristic function of A. However, in general

$$\int (f+g) d\mu_n \neq \int f d\mu_n + \int g d\mu_n$$

Theorem 5.1. *If $a_1, \ldots, a_n \in \mathbb{R}^+$, then the matrix $M_{ij} = [\min(a_i, a_j)]$ is positive semi-definite.*

Proof. We can assume without loss of generality that $a_1 \leq a_2 \leq \cdots \leq a_n$. We then write

$$M = \begin{bmatrix} a_1 & a_1 & a_1 & \cdots & a_1 \\ a_1 & a_2 & a_2 & \cdots & a_2 \\ a_1 & a_2 & a_3 & \cdots & a_3 \\ \vdots & & & & \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}\tag{5.2}$$

Subtracting the first column from the other columns gives the determinant

$$|M| = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_1 & a_2 - a_1 & a_2 - a_1 & \cdots & a_2 - a_1 \\ a_1 & a_2 - a_1 & a_3 - a_1 & \cdots & a_3 - a_1 \\ \vdots & & & & \\ a_1 & a_2 - a_1 & a_3 - a_1 & \cdots & a_n - a_1 \end{bmatrix} \tag{5.3}$$

We now prove by induction on n that

$$|M| = a_1(a_2 - a_1)(a_3 - a_2)\cdots(a_n - a_{n-1}) \tag{5.4}$$

For $n = 1$ we have $M = [a_1]$ and $|M| = a_1$ and for $n = 2$ we have

$$M = \begin{bmatrix} a_1 & a_1 \\ a_1 & a_2 \end{bmatrix}$$

and $|M| = a_1a_2 - a_1^2 = a_1(a_2 - a_1)$. Suppose the result (5.4) holds for $n - 1$ and let M have the form (5.2). Then $|M|$ has the form (5.3) so we have

$$|M| = a_1 \begin{bmatrix} a_2 - a_1 & a_2 - a_1 & a_2 - a_1 & \cdots & a_2 - a_1 \\ a_2 - a_1 & a_3 - a_1 & a_3 - a_1 & \cdots & a_3 - a_1 \\ a_2 - a_1 & a_3 - a_1 & a_4 - a_1 & \cdots & a_4 - a_1 \\ \vdots & & & & \\ a_2 - a_1 & a_3 - a_1 & a_4 - a_1 & \cdots & a_n - a_1 \end{bmatrix}$$

It follows from the induction hypothesis that

$$\begin{aligned} |M| &= a_1(a_2 - a_1)\,[(a_3 - a_1) - (a_2 - a_1)]\,[(a_4 - a_1) - (a_3 - a_1)] \\ &\qquad \cdots [(a_n - a_1) - (a_{n-1} - a_1)] \\ &= a_1(a_2 - a_1)(a_3 - a_2)\cdots(a_n - a_{n-1}) \end{aligned}$$

This completes the induction proof. Since $a_1 \le a_2 \le \cdots \le a_n$, we conclude that $|M| \ge 0$. Since all the principal submatrices of M have the form (5.2), they also have nonnegative determinants. Hence, M is positive semi-definite.

□

If $f\colon \Omega_n \to \mathbb{R}^+$, define the $2^n \times 2^n$ matrix $\widehat{f}$ given by

$$\widehat{f}_{ij} = \min\left[f(i), f(j)\right]$$

It follows from Theorem 5.1 that $\widehat{f}$ is positive semi-definite. For $f\colon \Omega_n \to \mathbb{R}$ we can write $f = f^+ - f^-$ in the canonical way where $f^+, f^- \geq 0$. Define the self-adjoint matrix $\widehat{f}$ by

$$\widehat{f}_{ij} = f_{ij}^{+\wedge} - f_{ij}^{-\wedge}$$

Applying (5.1) we have that

$$\int f d\mu_n = \sum_{i,j=0}^{2^n-1} \widehat{f}_{ij} D_{ij}^n \tag{5.5}$$

We might think of $\widehat{f}$ as the "observable" representing the "random variable" f. The next result shows that $\int f d\mu_n$ is then given by the usual quantum formula for the expectation of the observable $\widehat{f}$ in the "state" D^n.

Theorem 5.2. *For $f\colon \Omega_n \to \mathbb{R}$ we have that $\int f d\mu_n = \operatorname{tr}(\widehat{f} D^n)$.*

Proof. Applying (5.5), since $\widehat{f}$ is symmetric we have

$$\int f d\mu_n = \sum_{i,j} \widehat{f}_{ij} D_{ij}^n = \sum_{i,j} \widehat{f}_{ji} D_{ij}^n = \sum_j (\widehat{f} D^n)_{jj} = \operatorname{tr}(\widehat{f} D^n) \qquad \square$$

Corollary 5.3. *For any $A \in \mathcal{A}_n$ we have that $\mu_n(A) = \operatorname{tr}(\widehat{\chi}_A D^n)$.*

Proof. It follows from Theorem 5.2 that

$$\mu_n(A) = \int \chi_A d\mu_n = \operatorname{tr}(\widehat{\chi}_A D^n) \qquad \square$$

It is also interesting to note that $\widehat{\chi}_A = |\chi_A\rangle\langle\chi_A|$ so we can write $\mu_n(A) = \operatorname{tr}\left(|\chi_A\rangle\langle\chi_A| D^n\right) = \langle\chi_A|D^n|\chi_A\rangle$. More generally, we have

$$D_n(A,B) = \operatorname{tr}\left(|\chi_A\rangle\langle\chi_B| D^n\right) = \langle\chi_B|D^n|\chi_A\rangle \tag{5.6}$$

and $(A,B) \mapsto |\chi_A\rangle\langle\chi_B|$ is a positive semi-definite operator-bimeasure. That is, it is an operator-valued measure in each variable and $A_1, \ldots, A_m \subseteq \Omega_n$, $c_1, \ldots, c_m \subseteq \mathbb{C}$ imply that

$$\sum_{i,j} c_1 \overline{c_j} |\chi_{A_i}\rangle\langle\chi_{A_j}| \geq 0$$

Although $(f+g)^\wedge \neq \widehat{f}+\widehat{g}$ in general, the proof of the following lemma is straightforward.

Lemma 5.4. *If $f, g, h \colon \Omega_n \to \mathbb{R}$ have disjoint support, then*

$$(f+g+h)^\wedge = (f+g)^\wedge + (f+h)^\wedge + (g+h)^\wedge - \widehat{f} - \widehat{g} - \widehat{h}$$

Applying Lemma 5.4 and Theorem 5.2 gives the following result.

Corollary 5.5. *If $f, g, h \colon \Omega_n \to \mathbb{R}$ have disjoint support, then*

$$\begin{aligned}\int (f+g+h)d\mu_n = \int (f+g)d\mu_n + \int (f+h)d\mu_n + \int (g+h)d\mu_n \\ - \int f d\mu_n - \int g d\mu_n - \int h d\mu_n\end{aligned}$$

The next theorem can be used to simplify computations.

Theorem 5.6. *The eigenvalues of D^n are $1/2$ with multiplicity 2 and 0 with multiplicity $2^n - 2$. The unit eigenvectors corresponding to $1/2$ are ψ_0^n, ψ_1^n where*

$$\psi_0^n = \frac{1}{2^{(n-1)/2}} \begin{bmatrix} i^{c_n(0)} \\ 0 \\ i^{c_n(2)} \\ 0 \\ \vdots \\ i^{c_n(2^n-2)} \\ 0 \end{bmatrix}, \qquad \psi_1^n = \frac{1}{2^{(n-1)/2}} \begin{bmatrix} 0 \\ i^{c_n(1)} \\ 0 \\ i^{c_n(3)} \\ 0 \\ \vdots \\ 0 \\ i^{c_n(2^n-1)} \end{bmatrix}$$

Proof. Applying (2.3) we have for j odd that

$$D^n \psi_0^n(j) = 0 = \tfrac{1}{2}\, \psi_0^n(j)$$

and for j even that

$$\begin{aligned} D^n \psi_0^n(j) &= \frac{1}{2^{(3n-1)/2}} \sum \left\{ i^{[c_n(j)-c_n(k)]} i^{c_n(k)} \colon k \text{ even} \right\} \\ &= \frac{1}{2^{(3n-1)/2}}\, i^{c_n(j)} 2^{n-1} = \frac{1}{2} \frac{i^{c_n(j)}}{2^{(n-1)/2}} = \frac{1}{2}\, \psi_0^n(j) \end{aligned}$$

Hence, $D^n\psi_0 = \frac{1}{2}\psi_0$ and a similar argument shows that $D^n\psi_1 = \frac{1}{2}\psi_1$. Thus, $1/2$ is an eigenvalue with unit eigenvectors ψ_0^n, ψ_1^n. Now the kth column of D_n for $k > 0$ and k even is the vector

$$\begin{aligned}
&\left[\tfrac{1}{2^n}\ i^{-c_n(k)} i^{c_n(j)} p_{jk}, j = 0, 1, \ldots, 2^n - 1\right] \\
&\quad = \frac{i^{-c_n(k)}}{2^{(n+1)/2}} \left[\frac{1}{2^{(n-1)/2}} i^{c_n(j)} p_{jk}, j = 1, 2, \ldots, 2^n - 1\right] \\
&\quad = \frac{i^{-c_n(k)}}{2^{(n+1)/2}}\, \psi_0^n
\end{aligned}$$

Thus, the kth column of D^n for $k > 0$ and even is a multiple of ψ_0^n and similarly, the kth column of D^n for $k > 1$ and odd is a multiple of ψ_1^n. Hence, the range of D^n is generated by ψ_0^n and ψ_1^n. Thus, $\mathrm{Null}(D^n) = \mathrm{span}\left\{\psi_0^n, \psi_1^n\right\}^\perp$ so 0 is an eigenvalue of D^n with multiplicity $2^n - 2$. □

It follows from Theorem 5.6 that

$$D^n = \tfrac{1}{2}\,|\psi_0^n\rangle\langle\psi_0^n| + \tfrac{1}{2}|\psi_1^n\rangle\langle\psi_1^n| \tag{5.7}$$

Applying (5.6) and (5.7) gives

$$D_n(A, B) = \tfrac{1}{2}\,\langle\chi_A, \psi_0^n\rangle\langle\psi_0^n, \chi_B\rangle + \tfrac{1}{2}\,\langle\chi_A, \psi_1^n\rangle\langle\psi_1^n, \chi_B\rangle$$

and

$$\mu_n(A) = \tfrac{1}{2}\,\left|\langle\chi_A, \psi_0^n\rangle\right|^2 + \tfrac{1}{2}\,\left|\langle\chi_A, \psi_1^n\rangle\right|^2$$

Also, if $f\colon \Omega_n \to \mathbb{R}$, then by Theorem 5.2 we have

$$\int f d\mu_n = \mathrm{tr}(\widehat{f} D_n) = \tfrac{1}{2}\left\langle \widehat{f}\psi_0^n, \psi_0^n\right\rangle + \tfrac{1}{2}\left\langle \widehat{f}\psi_1^n, \psi_1^n\right\rangle \tag{5.8}$$

We close by computing some expectations. Let $f_n\colon \Omega_n \to \mathbb{R}^+$ be the random variable given by

$$f_n(\omega_i) = \text{ number of 1s in } \omega_i$$

The proof of the next result is similar to that of Lemma 4.5.

Lemma 5.7. *For $n \in \mathbb{N}$, $j = 0, 1, \ldots, 2^n - 1$, the function $f_n(j)$ satisfies*

$$f_{n+1}(j + 2^n) = f_n(j) + 1$$

Example 16. Since $f_1 = (0,1)$, it follows from Lemma 5.7 that $f_2 = (0,1,1,2)$, $f_3 = (0,1,1,2,1,2,2,3)$ and

$$f_4 = (0,1,1,2,1,2,2,3,1,2,2,3,2,3,3,4)$$

Example 17. Applying (5.8) we have that

$$\int f_1 d\mu_1 = 1/2, \quad \int f_2 d\mu_2 = 3/2, \quad \int f_3 d\mu_3 = 2$$

and

$$\int c_1 d\mu_1 = 1/2, \quad \int c_2 d\mu_2 = 3/2, \quad \int c_3 d\mu_3 = 3$$

Unfortunately, it appears to be difficult to find general formulas for $\int f_n d\mu_n$ and $\int c_n d\mu_n$.

This article is dedicated with affection to Josh Goldberg.

Research at Perimeter Institute for Theoretical Physics is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MRI.

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