To What Type of Logic Does the "Tetralemma" Belong?⋆

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Abstract

Although the so called tetralemma might seem to be incompatible with any recognized scheme of logical inference, its four alternatives arise naturally within the anhomomorphic logics which have been proposed in order to accommodate certain features of microscopic (i.e. quantum) physics. This suggests that the possibility of similar, "non-classical" logics might have been recognized in India at the time when Buddhism arose.

Considered from the standpoint of classical logic, the fourfold structure of the so called tetralemma (catuṣkoṭi) appears to be irrational, and modern commentators have struggled to explain its peculiar combination of alternatives, which at first sight appear to take the form, "A", "not-A", "A and not-A", "neither A nor not-A". (See for example [1] [2] [3] [4].) Such a combination does not accord with contemporary logic, and some authors have therefore concluded that the tetralemmas hail from an unknown logical system which rejects one or more cherished principles, like the "Law of the excluded middle" or the "Law of contradiction".† Others commentators have abandoned any attempt to make logical sense of the tetralemma, resorting rather to the hypothesis that its purpose is merely to illustrate that the propositions that occur in it are meaningless or ill-defined, like "The unicorn has black eyes", or (synesthesia aside) "The number seven is yellow".

⋆ E-print at `http://arXiv.org/abs/1003.5735 (math.LO)`

† See Appendix II.

According to Ruegg [3] the catuṣkoṭi form occurs frequently in the early Buddhist literature, and he provides (in translation) several examples, one being the question whether the world is

> finite, infinite, both finite and infinite, or neither finite nor infinite,

another being the question whether

> the world (of living beings) is eternal, not eternal, both eternal and not eternal, neither eternal nor not eternal.

Similarly, Shcherbatskoi [1] writes that, according to tradition, the founder of Buddhism refused to answer

> four questions regarding the beginning of the world, viz., there is a beginning, there is not, both, or neither.

Almost the same fourfold structure appears in another quotation from [3] (page 3):

> Entities of any kind are not ever found anywhere produced from themselves, from another, from both [themselves and another], and also from no cause.

In all of these examples, the four offerings are evidently intended as an exhaustive set of alternatives. And with the possible exception of the final tetralemma concerning causation, they all derive from a single "event" or "proposition", A. In the second example, for instance, A is the event of the world lasting forever, and if we abbreviate its negation as $\overline{A}$, then it seems as if we could express the four proffered alternatives as four propositions representable as: A; $\overline{A}$; A and $\overline{A}$; neither A nor $\overline{A}$. The difficulty of course is that the final two alternatives are self-contradictory when interpreted this way, leading one to wonder why they had to be included at all. (The tetralemma concerning causation might fall into a different category than the other three, depending on whether or not one assumes that every "entity" needs a cause or can have multiple causes. Nevertheless, it still clashes with classical logic in its denial that any one of the alternatives obtains. This case is discussed further in Appendix I.)

It seems clear that either the tetralemmas are unadorned nonsense or something other than classical logic is involved. In particular, one cannot explain away the evident contradictions merely by hypothesizing that questions about the beginning of the world or the infinity of existence are meant to be like questions about unicorns or about the color of the number seven. Even if that were the case, why would it not have sufficed to state the two alternatives, A and $\overline{A}$, after which one could either refuse to choose between them or deny both of them? Why did they need to be supplemented with two further alternatives, apparently trivial or lacking meaning altogether?

With respect to classical logic, this would have added nothing. But could it be that the questioners were aware of broader, more general logics, with respect to which the first two alternatives failed to cover all the relevant possibilities, and with respect to which the two additional alternatives were not in fact nonsense? If so, what might these other logics have been?

A possible answer comes from quantum mechanics, where the paradoxes that arise in the attempt to comprehend subatomic reality have given rise to non-classical logical systems that aim to reflect more faithfully the nature of the quantum world. In the early proposals of this sort, known collectively as "quantum logic", the laws for combining propositions via the connectives 'and', 'or', and 'not', are modified in such a way that the distributive law no longer holds [5] [6] [7]. More recently, a different type of logical framework has been put forward which modifies, not the connectives, but the rules of inference [8] [9] [10]. It is these *anhomomorphic* logics, I would suggest, that hold the key to understanding the catuṣkoṭi form.

In order to appreciate why this might be, one needs to distinguish carefully between what have been called "asserted" and "unasserted" propositions. That is, one needs to avoid confusing a proposition-in-itself with the affirmation or denial of that proposition.[⋆] Unaccompanied by either affirmation or denial, a proposition functions rather like a question or a predicate. It only indicates a possible event or state of affairs, without taking a stand on whether that event or state of affairs actually happens or obtains. For example, let A be the proposition "It rained all day yesterday". In itself, A tells us nothing about yesterday's weather; it only raises an implicit question. But if we then either deny or affirm A, we answer the implicit question, either negatively or positively, as the case may be.

⋆ In [11] Russel discusses this distinction at some length, contrasting his own attitude toward it with that of Frege.

In ordinary speech we rarely if ever make this distinction explicitly, though something rather like it is implicit in the grammar of conditional and counterfactual constructions. Perhaps for this reason, formal logic also seems to lack an agreed upon symbolism to portray the distinction. To remedy this lack, we can (following [8]) introduce a function ϕ that expresses affirmation or denial explicitly. Given a proposition † A, we can write $\phi(A) = 1$ in order to *affirm* A, and $\phi(A) = 0$ in order to *deny* it.

The distinction between a proposition per se and its affirmation or denial is closely related to the distinction between what, using a different language, might have been called "positive" and "negative" propositions. Consider the (unasserted) propositions, $A =$ "the electron is here" and $\overline{A} =$ "the electron is elsewhere". A statement like "I see the electron here" is in some sense positive, corresponding roughly to $\phi(A) = 1$ (i.e. to the affirmation of A). Similarly, "I see the electron there" is also positive, and corresponds roughly to $\phi(\overline{A}) = 1$. In contrast, a statement like "I don't see the electron here" is in some sense negative, corresponding roughly to $\phi(A) = 0$ (i.e. to the denial of A), while "I don't see the electron there" corresponds roughly to $\phi(\overline{A}) = 0$. Viewed in this manner, the affirmation of A is decoupled from the denial of $\overline{A}$, and four distinct alternatives arise, corresponding to those of the tetralemma.

Physically, the motivation for this type of de-coupling stems from the phenomenon of *interference*, in which, for example, an electron can act as if it were "in two places at once" or "in neither place separately but both together" [12]. Anhomomorphic logics accommodate this type of behavior by admitting, for instance, the possibility that both of the propositions, "the electron is here" and "the electron is elsewhere" might be false. In other words, one can have $\phi(A) = \phi(\overline{A})=0$ in such logics, just as one can have $\phi(A) =$

† From the standpoint of physics, the word "proposition" could be misleading, because it could suggest that a formula like $\phi(A) = 1$ refers more to a sentence in a language than to the real world. For this reason the word "event" (as in the world having a beginning) or "state of affairs" (as in the world being spatially finite) would be more apt than "proposition". Nevertheless, I have retained the latter word in deference to the usage that is traditional within formal logic. In the nomenclature of [8] the function ϕ is called a *coevent*. Using the "turnstile" notation, one could rewrite $\phi(A) = 1$ as $\vdash A$, but there does not seem to be an analogous notation for $\phi(A) = 0$.

$\phi(\overline{A})$=1.[♭] Thus, there is no necessary correlation in anhomomorphic logic between $\phi(A)$ and ϕ(not-A).

Consider now the sentence "The electron is not elsewhere" (or if you wish, "The world is not infinite"). With the possibility of an anhomomorphic logic in mind, we can see that its meaning is ambiguous. Perhaps it is simply formulating an unasserted proposition-in-itself, namely the above proposition $A =$ "the electron is here". On the other hand, perhaps it is trying to *deny* the complementary proposition $\overline{A}$, i.e. to express that $\phi(\overline{A}) = 0$. Or perhaps it actually means to *affirm* the proposition A, in which case one should render it as $\phi(A) = 1$. Accordingly, three different readings of our sentence are possible: (*i*) A, (*ii*) $\phi(\overline{A}) = 0$, (*iii*) $\phi(A) = 1$. In classical logic, there is no good reason to distinguish these meanings, but anhomomorphically it is crucial to do so. I would conjecture that most of the confusion over the tetralemma can be traced to a wrong resolution of this unrecognized ambiguity.

Take for definiteness the catuṣkoṭi about whether the world is finite, letting A be the proposition "the world is finite", and $\overline{A}$ the proposition "the world is infinite", which is the logical negation of A.[⋆] The question is how to interpret the alternatives that constitute the tetralemma.

If we interpret them as propositions-per-se, we will immediately land back in the usual difficulties, since even within anhomomorphic logic, the propositions, "both A and $\overline{A}$" and "neither A nor $\overline{A}$" are still nonsensical. (More precisely, both of them reduce to the "zero proposition", and including them as distinct alternatives adds nothing, since they could never be chosen as the answer.) Moreover, we can gain no comfort from any ambiguity in the symbolic form of these propositions, since for example, the propositions, $\overline{A} \wedge \overline{\overline{A}}$ and $\overline{A \vee \overline{A}}$ are strictly equivalent within anhomomorphic logic (and both equal

♭ Not every scheme of anhomomorphic logic recognizes all of these possibilities, however. For example in the *multiplicative scheme* described in [8] the combination $\phi(A) = \phi(\overline{A})$=1 cannot occur.

⋆ It is is not clear that an analysis in terms of complementary propositions A and $\overline{A}$ applies to the tetralemma about causation. See the appendices on this and related issues.

to zero).[†] (For this reason, the proposal in [4] to distinguish between these two forms of the fourth alternative would get nowhere within anhomomorphic logic.)

However if — in resolving the ambiguity pointed out above — we read the four alternatives differently, namely as the four possible combinations of affirmation and denial of the two complementary propositions A ("the world is finite") and $\overline{A}$ ("the world is infinite"), then the tetralemma makes perfect sense. In symbols, its four alternatives become then:

(1) $\phi(A) = 1$ and $\phi(\overline{A}) = 0$

(2) $\phi(A) = 0$ and $\phi(\overline{A}) = 1$

(3) $\phi(A) = 1$ and $\phi(\overline{A}) = 1$

(4) $\phi(A) = 0$ and $\phi(\overline{A}) = 0$

On this exegesis, the tetralemma format is simply the one that you adopt if you are seeking maximum generality, once you have admitted that the denial of a proposition A need not entail the affirmation of its logical complement $\overline{A}$, and vice versa. To the extent that Indian thinkers in the time of Gotama were aware of this possibility, they would naturally have phrased their questions in "tetralemmatic" form. (And of course, tradition would ensure that the form would persist, even if its original raison d'etre had been lost.)

But do we possess independent evidence that they were in fact aware of such more general forms of logic? Perhaps attention to the precise wording of the various catuṣkoṭi in their original languages would shed light on this question, but we also have at least one indication which appears to be relatively independent of that kind of delicate textual analysis. In reference [3], Ruegg writes that Indian grammarians recognized a type of "absolute" or "pure" negation that did not entail "affirmation of the contrary". Could this type of negation be what we have been expressing as $\phi(A) = 0$, and to which one should oppose the "propositional negation" that exchanges A with $\overline{A}$? And if so, could

[†] Here, I've used for brevity the common symbolism, $\vee$='or', $\wedge$='and'.

the introduction of two types of negation have been their way to decouple the denial of A from the affirmation of $\overline{A}$, and vice versa? [♭]

It would be very interesting to know what led people to recognize — if they did recognize — the possibility of an anhomomorphic logic over two millenia ago. They cannot have had access to the kind of technology that has led in modern times to quantum physics. Are there then other experiences that one could point to which were in fact available to them and to which anhomomorphic inference is more suited than homomorphic inference? If so, we might gain a better intuition for the microworld by ourselves paying more attention to those experiences.

Appendix I. A second-order application of anhomomorphic logic?

The analysis given in the body of the paper applies straightforwardly to all but the last example cited at the outset (about causation). If you accept it, it explains why these questions were posed in fourfold form. But if that were the whole story, then the answer to any given tetralemma should be a selection of one of the four mutually exclusive and exhaustive alternatives, (1)−(4), as the correct one. According to [13], the answers have indeed been like that in many cases. But in other cases, all four have been accepted or all four rejected, notably in the work of Nāgārjuna , who (unlike Gotama, who refused to answer at all) in his writings rejected all four of the alternatives [3].

Perhaps, as is commonly suggested, Nāgārjuna was simply trying to express a mystical rejection of analytical thought itself. However, it seems worth pointing out that anhomomorphic logic opens up another interpretation, perhaps consistent with the mystical one, but not really requiring it. Namely one can imagine that Nāgārjuna 's blanket denial represents a kind of "second order" application of anhomomorphic logic, one that reasons anhomomorphically, not just about "reality as such", but also about the logical processes employed to grasp that reality.

[♭] Westerhoff [13] also emphasizes the relevance of the two types of negation, *prasajya* and *paryudāsa* . In my interpretation, the former, or "absolute" negation corresponds to the denial of A, $\phi(A) = 0$, while the latter corresponds to the "propositional negation", $A \longleftrightarrow \overline{A}$.

When one is dealing with two propositions A and B (whether or not they are mutual negations so that $B = \overline{A}$), there will be four possible combinations of affirmation and denial of each, as explained above, i.e. four possible combinations of $\phi(A)$ and $\phi(B)$. When one is dealing with not two but four propositions, there will be 16 possible combinations, including the one that denies all four. Nāgārjuna could be opting for this last combination in respect of the four "second-order propositions" listed above as (1)−(4). It is natural to regard them as of second order since they are in some sense "propositions about propositions", speaking not just about "first order events", but about the values of ϕ on these "events". Thus, for example, one might try to symbolize the denial of alternative (3) by writing $\phi(\,\phi(A) = \phi(\,\overline{A}\,) = 1\,) \;=\; 0$.

Whether the tetralemma on causation also involves such a "denial at second order" seems uncertain. On one hand, it can be fit into the same mold as the others, if one treats "caused by self" and "caused by another" as mutual negations, and if one interprets its negative phrasing as the denial of all four alternatives. In that case, it would have the same character as the other examples, and Nargarjuna's blanket denial could again be interpreted either in the mystical or the "second order" manner.

However, such an exegesis would only be correct if one assumed that every event (or "entity" in the quoted translation) must have a unique cause. Without that assumption, the propositions A = "caused by self" and B = "caused by another" are not logical complements of each other, because the event in question might have multiple causes or no cause at all. In that case, it is not necessary (although it would still be possible) to interpret Nāgārjuna 's blanket denial in either of the above manners, because the naive symbolic-logic reading of the four alternatives is no longer problematic. That is, it could be that the alternatives are merely the perfectly respectable propositions: A; B; A and B; neither A nor B. Then one would have in this tetralemma a set of alternatives which happen to be four in number, not because they are formed from a single proposition A in the manner of the other tetralemmas, but because they represent all possible subsets of the two-element set {"caused by self", "caused by another"}. In that case, denying all the alternatives would only be a "first order" (though still anhomomorphic) activity, symbolizable as $\phi(A) = \phi(B) = \phi(A \wedge B) = \phi(\mathrm{not}(A \vee B)) = 0$.

Appendix II. Contradiction and excluded middle

It is often felt that the tetralemma's phrasing conflicts with the so-called laws of contradiction and of the excluded middle. Is such a conflict still present on the anhomomorphic interpretation? If not, this would furnish further evidence for the interpretation, inasmuch as — far from denying these two laws of logic — prominent schools of Buddhist logic seem to have embraced them, in particular the Madhyamaka school [3].

What these laws mean in modern terms seems to be not exactly settled, but to the extent that they govern the formation of compound propositions from simpler ones, they hold automatically in anhomomorphic logic. For example not-not-$A = A$ holds for any proposition A.

To the extent, however, that they are understood as laws of inference, the possibility of conflict does arise. For present purposes, let us take "excluded middle" to mean that 0 (false) and 1 (true) are the only values a proposition can assume, and let us take "non-contradiction" to mean that no proposition can take both of these values at once. Thus interpreted, both laws are honored in anhomomorphic logic, at least as it has developed so far. The former is obeyed because $\phi(A) = 0$ and $\phi(A) = 1$ are the only two values provided for, the latter simply because ϕ is assumed to be a function, as opposed to what used to be called a "multiply valued function".

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