

A Bell Inequality Analog in Quantum Measure Theory

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Abstract

One obtains Bell's inequalities if one posits a hypothetical joint probability distribution, or *measure*, whose marginals yield the probabilities produced by the spin measurements in question. The existence of a joint measure is in turn equivalent to a certain causality condition known as “screening off”. We show that if one assumes, more generally, a joint *quantal measure*, or “decoherence functional”, one obtains instead an analogous inequality weaker by a factor of $\sqrt{2}$. The proof of this “Tsirel'son inequality” is geometrical and rests on the possibility of associating a Hilbert space to any strongly positive quantal measure. These results lead both to a *question*: “Does a joint measure follow from some quantal analog of ‘screening off’?”, and to the *observation* that non-contextual hidden variables are viable in histories-based quantum mechanics, even if they are excluded classically.

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I. Introduction

Thinking of an experiment designed to test the Bell inequalities, we might picture to ourselves a source emitting a pair of silver atoms with correlated spins, and downstream, two Stern-Gerlach analyzers in spacelike separated regions, A and B . For each setting of the two analyzers one would obtain a set of $2 \times 2 = 4$ experimental probabilities (frequencies) corresponding to the four possible combinations of spin-up-or-down. By differently orienting one or both of the analyzers, one could similarly produce further sets of four experimental probabilities. A collection of probabilities obtained in this way, we will refer to as a *system of experimental probabilities*. The Bell inequality [1] (or more precisely its offspring, the Clauser-Horne-Shimony-Holt-Bell (CHSHB) inequality [2] [3]) pertains to such a system of experimental probabilities in the special case obtained by limiting each analyzer to only two possible settings (say a and a' for the A -analyzer, and b and b' for the B -analyzer).

Via a derivation that we recall below, the CHSHB inequality follows almost immediately from an assumption which we will express by saying that the given system of experimental probabilities *admits a joint probability distribution*. To clarify what this means, notice that, a priori, one has (with two settings each for the analyzers) four entirely distinct probability distributions, each living in its own four-element sample space $\Omega_{\alpha\beta} = \Omega_{\alpha} \times \Omega_{\beta}$, where α ranges over the settings a or a' of A , β ranges over the settings b or b' of B , and each space Ω_{α} , Ω_{β} is a binary sample space, corresponding to the two possibilities, spin-up/spin-down. To say that these probabilities admit a joint distribution means that one can merge the $\Omega_{\alpha\beta}$ into a single sample space

$$\widehat{\Omega} = \Omega_a \times \Omega_{a'} \times \Omega_b \times \Omega_{b'} \tag{1}$$

of $2^4 = 16$ elements, and that one can define on $\widehat{\Omega}$ a (not necessarily unique) probability distribution from which, for example, the probabilities for $\Omega_{ab} = \Omega_a \times \Omega_b$ follow on summing over the possible a' and b' outcomes. * That is, the separate distributions on the spaces $\Omega_{\alpha\beta}$

* That there are 16 experimental probabilities and 16 joint probabilities is merely a coincidence. The two numbers would differ if we generalized to particles of higher spin or considered more than two settings per analyzer.

can be recovered as *marginals* from a single probability distribution on the joint sample space $\widehat{\Omega}$.

In effect one is assigning a meaning to the so called “counterfactual” question, “What would I find if I could observe all four spin axes a, a', b and b' at once?” And — crucially — one is assuming that the distributions for the $\Omega_{\alpha\beta}$ (induced as marginals from the joint distribution on $\widehat{\Omega}$) are merely “*revealed*” but not *altered* by the particular way in which the analyzers are set, the “context” of the observation. For this reason, the assumption of a joint probability distribution is often alternatively described as the assumption of non-contextual “hidden variables” [4], and the violation of the CHSHB inequality is then described as an experimental refutation of such hidden variables theories. It is also described as a refutation of “local causality” because a condition of that type implies the existence of non-contextual hidden variables.

Thus far, however, the implicit context of our discussion has been entirely classical, and one may wonder to what extent the relationships we have just reviewed carry over to the quantum case. It might seem that this question is ill posed, for the lack of a quantal analog of the notion of joint probability distribution. However, if one views quantum mechanics from a “histories” standpoint, then it is natural to regard it as a kind of generalized theory of probability or *measure* (probability being realized mathematically in terms of the concept of measure). Indeed, one can delineate a hierarchy of such generalized measure theories [5] in which classical stochastic theories comprise the first level of the hierarchy and unitary quantum theories — suitably interpreted — are included in the second level. (See also [6].)

Within this second level, the level of “quantal measures” or “decoherence functionals” (we use the terms interchangeably), one has a notion of joint quantal measure, in direct analogy to the notion of joint probability distribution. We will see that just as the assumption that the experimental probabilities admit a joint *classical* measure leads to the CHSHB inequality, so the assumption that they admit a joint *quantal* measure leads, almost as directly, to an analogous but weaker constraint known as the Tsirel’son inequality [7]. The main result of this paper, then, is that the latter inequality can be understood as a direct analog of the CHSHB inequality if one adopts a histories formulation of quantum

mechanics. Such a formulation also leads to a geometrical proof of the inequality that, we believe, has some independent interest in its own right.[†]

Of course, the connection between the CHSHB inequality and the existence of a joint measure is far from the whole story in the classical case, because the strongest support for the latter assumption usually comes from considerations of causality and/or locality.^b Of particular importance in this connection is the condition known variously as “local causality” [10], “stochastic Einstein locality” [11], or “classical screening off”.

The condition of “screening off” on the classical measure asserts that events in causally unrelated regions of spacetime become independent (are “screened off” from one another) when one conditions on a complete specification of the history in their mutual causal past. As shown by Fine [4], the derivation of the CHSHB inequality from screening off can be viewed as a two-step process. First one goes from screening off to the existence of a joint probability distribution, and then from the latter to the inequality. (The converse implications are also valid [4].) The violation in nature of the CHSHB inequality is thus also a violation of classical screening off. Usually this is described as a “failure of locality”, but because screening off is above all a condition of relativistic *causality*, it might be more appropriate to rather characterize violation of the CHSHB inequality as a “failure of (classical) causality”.

Should quantum mechanics, then, be thought of as nonlocal, acausal, or both — or is there a sense in which it is neither if seen from an appropriate vantage point? We would have liked, in the present paper, to provide such a vantage point by showing that the classical threefold equivalence among screening off, the existence of a joint probability measure, and the CHSHB inequality reproduces itself at a higher level (namely level two) as

[†] A related result, which identifies “quantum Bell inequalities” which are necessary conditions for a set of two-qubit states to be the reduced states of a mixed state of three qubits, appears in [8].

^b Not all authors distinguish between these concepts, but we try to do so consistently here, *cf.* [9]. By *locality* we mean the failure of physical influences to “jump over regions of spacetime”, and by *causality* (in the sense of relativistic causality) we mean their failure to act outside the future light cone. For example, a theory containing “tachyons” might be local without being causal.

a relationship among *quantal* screening off, the existence of a joint decoherence functional, and the Tsirel'son inequality (which of course is *not* violated by quantum mechanics).

The proof that a joint quantal measure implies the Tsirel'son inequality accomplishes this in part, but we are unable to complete the story in all generality because we lack a fully convincing formulation of quantal screening off. Nevertheless, we will suggest a candidate condition that closely resembles its classical analog, and that is formally valid in relativistic quantum field theory. We will be able to prove that any system of experimental probabilities that admits a joint decoherence functional also admits a model which obeys this screening off condition; but the converse eludes us, and so we cannot yet assert that screening off is fully equivalent to a joint measure in the quantal case. We *will* show, however, that a causality assumption inherent in standard unitary quantum theory, namely the commuting of spacelike separated operators, does imply the existence of a joint measure. This provides a kind of converse and shows in particular how our proof of the Tsirel'son inequality can be founded on a recognizable causality condition.

II. Quantum mechanics as quantum measure theory

We briefly summarize the hierarchy of generalized measure theories described in more detail in [5] [12] [13].

In a generalized measure theory, there is a sample space Ω of possibilities for the system in question. Normally these are to be thought of as “fine grained histories”, meaning as complete a description of physical reality as is conceivable in the theory, *e.g.* for n -particle mechanics a history would be a set of n trajectories, and for a scalar field theory, a history would be a field configuration on spacetime. Predictions about the system — the dynamical content of the theory — are to be gleaned, in some way or another, from a (generalized) measure μ on Ω (strictly, on some suitable class of “measurable” subsets of Ω , but we will gloss over this technicality here).

Given μ (a non-negative real valued set function), we can construct the following series of symmetric set functions:

$$I_1(X) \equiv \mu(X)$$

$$I_2(X, Y) \equiv \mu(X \sqcup Y) - \mu(X) - \mu(Y)$$

$$I_3(X, Y, Z) \equiv \mu(X \sqcup Y \sqcup Z) - \mu(X \sqcup Y) - \mu(Y \sqcup Z) - \mu(Z \sqcup X) + \mu(X) + \mu(Y) + \mu(Z)$$

and so on, where X, Y, Z , etc. are disjoint subsets of Ω , as indicated by the symbol ‘ \sqcup ’ for disjoint union.

A measure theory of level k is one which satisfies the sum rule $I_{k+1} = 0$. It is known that this condition implies that all higher sum rules are automatically satisfied, viz. $I_{k+n} = 0$ for all $n \geq 1$. A level 1 theory is thus one in which the measure satisfies the usual Kolmogorov sum rules of classical probability theory, classical Brownian motion being a good example. A level 2 theory is one in which the Kolmogorov sum rules may be violated but I_3 is nevertheless zero. Unitary quantum mechanics satisfies this condition and is an example of a level 2 theory — which we dub therefore “quantum measure theory” in general.

The existence of a normalized quantum measure on Ω is equivalent to the existence of a *decoherence functional* $D(X; Y)$ of pairs of subsets of Ω satisfying: *

- (i) Hermiticity: $D(X; Y) = D(Y; X)^*$, $\forall X, Y$;
- (ii) Additivity: $D(X \sqcup Y; Z) = D(X; Z) + D(Y; Z)$, $\forall X, Y, Z$ with X and Y disjoint;
- (iii) Positivity: $D(X; X) \geq 0$, $\forall X$;
- (iv) Normalization: $D(\Omega; \Omega) = 1$.

The relationship between the quantal measure and the decoherence functional is

$$\mu(X) = D(X; X). \tag{2}$$

Unless otherwise stated, we will always assume that D satisfies in addition to (iii) the condition of *strong positivity*, which states that for any finite collection of (not necessarily disjoint) subsets X_1, X_2, \dots, X_n of Ω , the $n \times n$ Hermitian matrix $M_{ij} = D(X_i; X_j)$ is positive semidefinite (it has no negative expectation values). The decoherence functional of

* The quantity $D(X; Y)$ is interpretable as the quantum interference between two sets of histories in the case when they are disjoint. Notice from (2) that μ determines only the real part of D . (The imaginary part of D influences how smaller systems combine to form bigger ones. It also may affect the consistency/decoherence conditions one wishes to impose. These issues will be discussed in greater depth elsewhere.)

ordinary unitary quantum mechanics, for example, is strongly positive. Strong positivity is a powerful requirement because it implies in general that there is a Hilbert space associated with the quantum measure, which turns out to be the standard Hilbert space in the case of unitary quantum mechanics [14] [15]. Decoherence functionals which merely satisfy condition (iii) above are termed “weakly positive”.

In this paper, we will not enter into the general question of how to interpret the quantum measure. One set of ideas for doing so goes by the name of “consistent histories” or “decoherent histories” and attempts in effect to reduce the quantal measure to a classical one by the imposition of decoherence conditions [16] [17] [18] [19]. A different attempt at an interpretation, based on the notions of “preclusion” and correlation, may be found in [12]. For our purposes in this paper, it will suffice to assume, where macroscopic measuring instruments are concerned, that distinct “pointer readings” do not interfere (they “decohere”), and that their measures can be interpreted as probabilities in the sense of frequencies.

In the sequel, we adopt a usage that seems particularly suitable for a histories-based measure theory. We use the terminology “an *event* in spacetime region A ” to refer to a subset $a \subseteq \Omega$ such that the criterion which determines whether or not a history γ belongs to a refers only to the properties of γ within A (*e.g.* if γ is a field then its restriction to A is supposed to be enough information to determine whether $\gamma \in a$).[†]

We will also assume that all of our sample spaces Ω are finite, so that integrals may be written as sums. Among other things, this lets us avoid the main technical complications in the definition of conditional probability.

[†] The term “event” is standard in probability theory for a subset of Ω . A subset of histories defined by some common property is termed a “coarse grained history” in the standard parlance of consistent histories quantum theory. In this language, an event in A is therefore a coarse grained history defined by a coarse graining according to properties local to A .

III. Two inequalities, classical and quantal

Classical case

We rehearse the proof of the CHSHB inequality at level one in the hierarchy of generalized measures — *i.e.* at the classical level. The experimental context will be that described in the Introduction. In formalising it, however, one faces a choice. Namely, one must decide whether or not to include in the analysis, random variables corresponding to the instrument settings. If one excludes such variables, then one need deal only with the sample spaces described in the Introduction: the four spaces $\Omega_{\alpha\beta}$, the four spaces Ω_α and Ω_β , and the joint sample space $\widehat{\Omega}$. (Recall that in our notation, $\alpha = a$ or a' , and $\beta = b$ or b' , the instrument settings in regions A and B .) On the other hand, if one includes the instrument settings as variables, then one necessarily deals with a larger sample space Ω . We have chosen to follow the second approach (which arguably is “more fully intrinsic”, in keeping with the philosophy of generalized quantum mechanics as a theory of closed systems), and consequently our discussion will attribute probabilities not only to the possible outcomes with a given experimental arrangement, but also to the possible experimental arrangements themselves. Nevertheless, the following may also be read consistently as if the first, more “minimalist” approach had been adopted, since, *mutatis mutandis*, the proofs take the same form in both cases. (One who feels uncomfortable attributing quantitative probabilities to instrument settings may thus refrain from doing so.) The essential difference between the two approaches is that in the “minimalist” reading, conditional probabilities like $\text{Prob}(\textit{outcome}|\textit{setting})$ must be understood as primitive objects; they cannot be resolved into ratios of conditional probabilities like $\text{Prob}(\textit{outcome} \cap \textit{setting})/\text{Prob}(\textit{setting})$.

Consider a sample space, Ω , of histories defined on a “substratum” possessing a background causal structure (a spacetime, for example, or a causal set). Let A and B be two spacelike separated regions of the substratum and denote their causal pasts by $J^-(A)$ and $J^-(B)$ (where $J^-(A)$ contains A itself). We are interested in the usual EPRB setup in which there is a range of possible choices (to be made “essentially freely”) of *settings* of some experimental apparatus in A and similarly in B . For example, this range might be the possible directions of the magnetic field in a Stern-Gerlach apparatus for spin measurements. For each setting in A , the outcome of the measurement is either $+1$ or -1 . In

the standard example this would be the measured value of the spin (multiplied by $2/\hbar$) in the set direction.

Let M_A denote the set of possible settings of the experimental apparatus in A . (As mentioned in the Introduction, we limit ourselves to two settings, $M_A = \{a, a'\}$.) Each element of M_A is, in our technical sense, an event in A (or more generally in $J^-(A) \cap J^-(B)^c$, where the superscript c denotes complementation), namely, that subset of Ω containing those histories in which the corresponding experimental setting is made. The elements of M_A are disjoint. For each element, $\alpha \in M_A$, let $\alpha_{\pm 1}$ denote the possible outcomes of the measurement with setting α , so that $\alpha = \alpha_{+1} \sqcup \alpha_{-1}$ where α_{+1} (α_{-1}) is the set of histories in which outcome $+1$ (-1) obtains. Similarly, M_B is the set of two possible B -measurements, $\{b, b'\}$; each element of M_B is an event in B (or $J^-(B) \cap J^-(A)^c$); and for each $\beta \in M_B$, $\beta_{\pm 1}$ are the possible outcomes of the measurement, with $\beta = \beta_{+1} \sqcup \beta_{-1}$.

The “law of motion” of the underlying stochastic process is assumed to be given by a classical probability measure μ on Ω , and an expression like $\mu(\alpha_i \cap \beta_j | \alpha \cap \beta)$ will denote the probability of outcomes α_i and β_j , *conditional on* the settings being α and β . Since we are imagining all our sample spaces as finite, a conditional probability $\mu(x|y) = \mu(x \cap y) / \mu(y)$ is only really meaningful when $\mu(y) > 0$. In the contrary case, one might define it to be zero, since $\mu(y) = 0 \Rightarrow \mu(x \cap y) = 0$ (albeit not when μ is quantal!), but for present purposes, it will prove more convenient to adopt the convention that $\mu(x|y)$ is simply *undefined* when $\mu(y)$ vanishes.

Let $\widehat{\Omega}$ be the sixteen-element sample space (1) labelled by the (16 possible values of the) quadruple of binary variables $(a_i, a'_{i'}, b_j, b'_{j'})$, each of which takes values ± 1 . (For brevity we will write $(a_i, a'_{i'}, b_j, b'_{j'}) \equiv (i' i' j j')$ where there is no risk of confusion.) In the Introduction, we called the sixteen numbers $\mu(a_i \cap b_j | a \cap b)$, $\mu(a'_{i'} \cap b_j | a' \cap b)$, $\mu(a_i \cap b'_{j'} | a \cap b')$, $\mu(a'_{i'} \cap b'_{j'} | a' \cap b')$, a “system of experimental probabilities”, and we agreed to say that these numbers *admit a joint probability distribution* if and only if there exists a classical measure, $\widehat{\mu}$ on $\widehat{\Omega}$, such that

$$\mu(a_i \cap b_j | a \cap b) = \sum_{i' j'} \widehat{\mu}(i' i' j j') , \quad (3)$$

and similarly for every other (α, β) pair. It is now easy to prove the CHSHB inequality.

Theorem 1 Let Ω and μ be as described above and assume that the resulting system of experimental probabilities admits a joint probability distribution $\hat{\mu}$ on $\hat{\Omega}$ satisfying the condition (3) on its marginals. Define the correlation functions

$$X(\alpha, \beta) \equiv \sum_{ij} i \cdot j \cdot \mu(\alpha_i \cap \beta_j | \alpha \cap \beta) \quad (4)$$

for $\alpha = a, a', \beta = b, b'$. Then

$$| X(a, b) + X(a', b) + X(a, b') - X(a', b') | \leq 2 . \quad (5)$$

(The pattern is three plus signs and a minus. It doesn't matter where one puts the minus sign.)

PROOF It suffices to prove the inequality without the absolute value signs, as one sees by reversing the signs of the B -outcomes. By assumption there exists a measure $\hat{\mu}$ on the sample space $\hat{\Omega}$ of quadruples $(ii'jj')$ whose marginals agree with μ on each (α, β) pair. Therefore

$$\begin{aligned} X(a, b) &= \sum_{ij} i \cdot j \cdot \mu(a_i \cap b_j | a \cap b) \\ &= \sum_{ii'jj'} i \cdot j \cdot \hat{\mu}(ii'jj') , \end{aligned}$$

with similar formulas for $X(a', b)$, $X(a, b')$ and $X(a', b')$. But for any of the possible values of i, i', j, j' we have $ij + i'j + ij' - i'j' = (i + i')j + (i - i')j' \leq 2$, since one of the two parentheses must vanish in every case. The weighted average with respect to $\hat{\mu}$ of this combination of i 's and j 's is therefore also less than or equal to 2; hence

$$X(a, b) + X(a', b) + X(a, b') - X(a', b') = \sum_{ii'jj'} (ij + i'j + ij' - i'j') \hat{\mu}(ii'jj') \leq 2 .$$

QED

Quantal case

At level two we have the same setup as before: a sample space Ω including setting events $a, a', b, b', etc.$; and we use the same notation, in particular $\alpha = a$ or a' and $\beta = b$ or b' . But now we have on Ω a *quantal* measure μ and the associated decoherence functional D . We are considering the situation in which the events $\alpha_i \cap \beta_j$ correspond to the readings

(and settings) of macroscopic instruments, and so, as announced earlier, we will assume that the quantal measure μ of any one of these macroscopic “instrument events” can be interpreted as an experimental probability (*i.e.* a frequency). Having done so, we can form *conditional probabilities* in the standard manner, as illustrated by the definition of the $p(\alpha_i, \beta_j)$ in equation (7) below. The correlators $X(\alpha, \beta)$ are then definable exactly as in the classical case [equation (4)]. (Notice that we have not attempted to extend the notion of conditional probability outside the setting of classical (level 1) measure theory. To our knowledge, there is, unfortunately, no established notion of “conditional quantal measure” or “conditional decoherence functional”, of which classical conditional probability would be a special case.)^b

Notice that the identification of $\mu(Y) = D(Y; Y)$ as a probability-*qua*-frequency is only consistent over the whole algebra of instrument events Y if we assume that neither distinct instrument settings nor distinct outcomes for given settings interfere with one another.^{*} In other words, we must assume that

$$D(\alpha_i \cap \beta_j ; \alpha_k \cap \beta_l) = \mu(\alpha_i \cap \beta_j) \delta_{ik} \delta_{jl} , \quad (6)$$

$\forall \alpha, \beta$; and we also assume that all remaining such off-diagonal values of D vanish, for example, $D(a_i \cap b_j ; a'_k \cap b_l) = 0$.

DEFINITION We denote as the *experimental probabilities* the sixteen numbers,

$$p(\alpha_i, \beta_j) = \mu(\alpha_i \cap \beta_j | \alpha \cap \beta) = \frac{\mu(\alpha_i \cap \beta_j)}{\mu(\alpha \cap \beta)} . \quad (7)$$

^b The need for a quantal generalization of conditional probability arises in the following only because the experimental probabilities we work with are conditioned on specific instrument settings. For present purposes, it thus would not arise at all in the alternative, “minimalist approach” mentioned earlier. However, even in such a framework, the need would return as soon as one had to condition on the specific *results* of observations or other processes.

^{*} This consistency condition is what gives the “consistent histories” interpretation its name. But there, it is raised to the level of a principle.

DEFINITION The experimental probabilities $p(\alpha_i, \beta_j)$ admit a joint quantal measure iff there exists a decoherence functional \widehat{D} on $\widehat{\Omega}$ such that its *marginals* agree with (7) for each of the four (α, β) pairs (*i.e.* for each of the four possible instrument settings):

$$\widehat{D}_{ab}(ij; kl) \equiv \sum_{i'j'k'l'} \widehat{D}(i'j'j'; kk' ll') = p(a_i, b_j) \delta_{ik} \delta_{jl} \quad (8ab)$$

for $(\alpha, \beta) = (a, b)$;

$$\widehat{D}_{a'b}(i'j; k'l) \equiv \sum_{ij'kl'} \widehat{D}(ii'jj'; kk' ll') = p(a'_i, b_j) \delta_{i'k'} \delta_{jl} \quad (8a'b)$$

for $(\alpha, \beta) = (a', b)$; and similarly for $(\alpha, \beta) = (a, b')$ and $(\alpha, \beta) = (a', b')$.

REMARK Our use of the word “marginals” here is in obvious analogy to its use in classical measure theory, where, given that $\Omega = \Omega_1 \times \Omega_2$, a marginal probability distribution on Ω_2 is one induced from Ω by summing over Ω_1 . Similarly here, the joint decoherence functional \widehat{D} on $\widehat{\Omega}$ induces marginal decoherence functionals $\widehat{D}_{\alpha\beta}$ (and hence marginal quantal measures $\widehat{\mu}_{\alpha\beta}$) on all the (α, β) pairs, as illustrated in (8ab) and (8a'b).

Observe that the matching-conditions (8) on the marginals require more than just agreement with the 16 probabilities $p(\alpha_i, \beta_j)$. They also entail the vanishing of the 24 off-diagonal elements $\widehat{D}_{\alpha\beta}(ij; kl)$ with $i \neq k$ or $j \neq l$. Notice on the other hand, that they do not refer to any marginals that would involve interference between distinct instrument settings.

We will see that the Tsirel'son inequality follows from the existence of such a joint quantal measure. However, in order to demonstrate this, we will need to apply to \widehat{D} a certain basic construction via which any strongly positive decoherence functional gives rise to a Hilbert space [14] [15].

Hilbert space from (strongly positive) quantal measure

Consider the vector space \mathfrak{H}_1 which consists of all formal linear combinations of the sixteen four-bit strings, $(ii'jj')$, $i, i', j, j' = \pm 1$. Let $[ii'jj']$ denote a general basis vector of \mathfrak{H}_1 . That \widehat{D} is strongly positive means that it induces a (possibly degenerate) Hermitian inner product on \mathfrak{H}_1 given by:

$$\langle [ii'jj'], [kk'll'] \rangle = \widehat{D}(ii'jj'; kk'll') .$$

In general, \mathfrak{H}_1 is not a Hilbert space because it contains vectors with zero norm. To form a true Hilbert space \mathfrak{H} take the quotient of \mathfrak{H}_1 by the vector subspace \mathfrak{H}_0 of zero norm states: $\mathfrak{H} = \mathfrak{H}_1/\mathfrak{H}_0$. Denote by $|ii'jj'\rangle$ the vector in \mathfrak{H} that corresponds to $[ii'jj'] \in \mathfrak{H}_1$. (Regarding members of $\mathfrak{H}_1/\mathfrak{H}_0$ as equivalence classes, we can describe $|ii'jj'\rangle$ as the set of vectors in \mathfrak{H}_1 that differ from $[ii'jj']$ by vectors of zero norm.) Plainly, the vectors $|ii'jj'\rangle$ span \mathfrak{H} ; and we have^b

$$\widehat{D}(ii'jj';kk'll') = \langle ii'jj'|kk'll'\rangle. \quad (9)$$

This relationship will let us convert the correlators $X(\alpha, \beta)$ into inner products of vectors in \mathfrak{H} , the key step in our proof of Theorem 2.

Correlators as inner products in Hilbert space: proof of Theorem 2

In this subsection, we state and prove our main result as a theorem. We assume that the experimental probabilities admit a joint quantal measure given by the decoherence functional \widehat{D} on $\widehat{\Omega}$, as specified in equations (8), and we denote by $\widehat{\mu}$ the corresponding generalized measure given by the diagonal elements of \widehat{D} , as in equation (2).

Lemma 3.1 Let $|a\rangle \in \mathfrak{H}$ be defined by

$$|a\rangle = \sum_{ii'jj'} i \cdot |ii'jj'\rangle \quad (10)$$

and similarly for $|a'\rangle$, $|b\rangle$ and $|b'\rangle$. Then $\langle a|a\rangle = \langle b|b\rangle = \langle a'|a'\rangle = \langle b'|b'\rangle = 1$, and for any of the four possible pairings of $\alpha = a, a'$ with $\beta = b, b'$, we have

$$\langle \alpha|\beta\rangle = X(\alpha, \beta) \equiv \sum_{ij} i \cdot j \cdot p(\alpha_i, \beta_j) \quad (11)$$

PROOF We give the proof for $\alpha = a, \beta = b$, the other three cases being strictly analogous. Using equations (8), (9) and (10), we replace the sum over diagonal terms in $X(a, b)$ by

^b For related observations see [20] [21] [22].

the full sum:

$$\begin{aligned}
X(a, b) &= \sum_{ij} i \cdot j \cdot p(a_i, b_j) \\
&= \sum_{ijkl} i \cdot l \cdot p(a_i, b_j) \delta_{ik} \delta_{jl} \\
&= \sum_{ijkl} i \cdot l \cdot \widehat{D}_{ab}(ij; kl) \\
&= \sum_{ijkl} i \cdot l \cdot \sum_{i'j'k'l'} \widehat{D}(ii'jj'; kk'll') \\
&= \sum_{ii'jj'kk'll'} i \cdot l \cdot \widehat{D}(ii'jj'; kk'll') \\
&= \sum_{ii'jj'kk'll'} i \cdot l \cdot \langle ii'jj' | kk'll' \rangle \\
&= \langle a | b \rangle .
\end{aligned}$$

We must also prove that the vectors $|\alpha\rangle$, $|\beta\rangle$ have unit norm. Let us prove for example that $\langle a | a \rangle = 1$. To that end, define the vectors

$$|a\pm\rangle \equiv \sum_{i'jj'} | \pm 1 i' j j' \rangle$$

and note that $\langle a+ | a- \rangle = 0$ by (8ab) with $i = +1$, $k = -1$. Then

$$|a\rangle = |a+\rangle - |a-\rangle$$

and we have

$$\begin{aligned}
\langle a | a \rangle &= \langle a+ | a+ \rangle + \langle a- | a- \rangle - \langle a+ | a- \rangle - \langle a- | a+ \rangle \\
&= \langle a+ | a+ \rangle + \langle a- | a- \rangle + \langle a+ | a- \rangle + \langle a- | a+ \rangle
\end{aligned}$$

This last line is $\widehat{D}(\widehat{\Omega}; \widehat{\Omega})$, which is 1 by our assumption of normalization. QED

We are now ready to prove our main result, that any set of experimental probabilities which admits a joint quantal measure must respect the Tsirel'son inequality:

Theorem 2 If there exists a strongly positive joint decoherence functional \widehat{D} on $\widehat{\Omega}$ whose marginals agree with D — meaning equations (8) hold — then

$$| X(a, b) + X(a', b) + X(a, b') - X(a', b') | \leq 2\sqrt{2} . \quad (12)$$

REMARK Instead of saying that the marginals “agree with D ”, we could equally well have said that they “are diagonal and yield the experimental probabilities $p(\alpha_i, \beta_j)$ ”. This expresses the theorem in a more self-contained form.

PROOF As before, it suffices to prove (12) without the absolute value signs. Write

$$Q \equiv X(a, b) + X(a', b) + X(a, b') - X(a', b') \quad (13)$$

which has a “logical” maximum value of 4. By the previous lemma, we have

$$\begin{aligned} Q &= \langle a|b \rangle + \langle a'|b \rangle + \langle a|b' \rangle - \langle a'|b' \rangle \\ &= (\langle a| + \langle a'|) |b \rangle + (\langle a| - \langle a'|) |b' \rangle . \end{aligned} \quad (14)$$

Since $|b \rangle$ and $|b' \rangle$ are unit vectors, Q is maximized when $|b \rangle$ is parallel to $|a \rangle + |a' \rangle$ and $|b' \rangle$ is parallel to $|a \rangle - |a' \rangle$. Hence

$$Q \leq \| |a \rangle + |a' \rangle \| + \| |a \rangle - |a' \rangle \| ,$$

whence $Q \leq 2\sqrt{2}$ by the following simple lemma. QED

Lemma 3.2 If u and v are vectors of unit length then $\|u + v\| + \|u - v\| \leq \sqrt{8} = 2\sqrt{2}$.

PROOF Let $S = \|u + v\| + \|u - v\|$ and write $\xi = \text{Re}\langle u|v \rangle$. Then $\|u \pm v\|^2 = \langle u \pm v|u \pm v \rangle = (1 + 1 \pm 2\xi) = 2 \pm 2\xi$. Hence

$$\begin{aligned} S^2 &= \|u + v\|^2 + \|u - v\|^2 + 2\|u - v\|\|u + v\| \\ &= (2 + 2\xi) + (2 - 2\xi) + 2\sqrt{(2 + 2\xi)(2 - 2\xi)} \\ &= 4 + 2\sqrt{4 - 4\xi^2} \\ &\leq 4 + 2\sqrt{4} = 8 \\ &\Rightarrow S \leq \sqrt{8} . \end{aligned}$$

QED

An example: saturating the bound

To illustrate some of the above, consider the familiar quantum mechanical setup leading to maximal violation of Bell’s inequalities (5), which is known to produce a system of experimental probabilities that saturates the Tsirel’son bound (12). The mathematics involved

in this situation produces a joint decoherence functional that also is on the boundary of the convex set of strongly positive decoherence functionals. We can use this to conclude that by itself, weak positivity of the quantal measure (*i.e.* the condition $D(X; X) \geq 0$ on the decoherence functional) is insufficient to imply the bound (12).

We assume we have two spin-half particles in a singlet state and each particle heads off to either region A or region B where Alya and Bai, respectively, are waiting to make measurements on the particles. Alya sets her apparatus to measure the spin in directions \mathbf{a} or \mathbf{a}' and Bai in directions \mathbf{b} or \mathbf{b}' where \mathbf{a} , \mathbf{a}' , \mathbf{b} and \mathbf{b}' are now unit vectors in three dimensional space satisfying

$$\begin{aligned}\mathbf{a} \cdot \mathbf{a}' &= \mathbf{b} \cdot \mathbf{b}' = 0 \\ \mathbf{a} \cdot \mathbf{b} &= \mathbf{a} \cdot \mathbf{b}' = \mathbf{a}' \cdot \mathbf{b} = -\mathbf{a}' \cdot \mathbf{b}' = \frac{1}{\sqrt{2}} .\end{aligned}$$

It is interesting that when calculating the quantity Q as given by ordinary quantum mechanics in this setup we obtain

$$\begin{aligned}X(\mathbf{a}, \mathbf{b}) &+ X(\mathbf{a}', \mathbf{b}) + X(\mathbf{a}, \mathbf{b}') - X(\mathbf{a}', \mathbf{b}') \\ &= (\mathbf{a} + \mathbf{a}') \cdot \mathbf{b} + (\mathbf{a} - \mathbf{a}') \cdot \mathbf{b}' ,\end{aligned}$$

which is exactly the same expression (14) as arose in the general proof of (12), only here we have ordinary vectors in \mathbb{R}^3 instead of vectors in Hilbert space.

In the EPRB setup we have a 4-dimensional Hilbert space \mathfrak{H} which is a tensor product of two qubit Hilbert spaces \mathfrak{H}_A and \mathfrak{H}_B , and $|\psi\rangle = (|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B)/\sqrt{2}$ is the singlet state. On \mathfrak{H}_A we have Pauli matrices $\boldsymbol{\sigma}$ and on \mathfrak{H}_B we have Pauli matrices $\boldsymbol{\rho}$, from which we can form projection operators.

We will form the decoherence functional from the expectation value in the singlet state of strings of projectors onto the several values of the two spins in the four directions, \mathbf{a} , \mathbf{a}' , \mathbf{b} , \mathbf{b}' . Specifically, let us form the decoherence functional using the strings of projectors appropriate to the results: “Alya finds the spin to be $i\hbar/2$ in the \mathbf{a} direction *and then*

$i\hbar/2$ in the \mathbf{a}' direction, and Bai finds $j\hbar/2$ in the \mathbf{b} direction *and then* $j'\hbar/2$ in the \mathbf{b}' direction.” For this we need the projectors

$$\begin{aligned} P_i^a &\equiv \frac{1}{2}(1 + i \mathbf{a} \cdot \boldsymbol{\sigma}) \\ P_{i'}^{a'} &\equiv \frac{1}{2}(1 + i' \mathbf{a}' \cdot \boldsymbol{\sigma}) \\ P_j^b &\equiv \frac{1}{2}(1 + j \mathbf{b} \cdot \boldsymbol{\rho}) \\ P_{j'}^{b'} &\equiv \frac{1}{2}(1 + j' \mathbf{b}' \cdot \boldsymbol{\rho}). \end{aligned}$$

From these and the initial singlet state $|\psi\rangle$ we can construct the following decoherence functional that is strongly positive and decoheres on all (α, β) pairs:

$$\widehat{D}(ii'jj'; kk'll') = \langle \psi | P_k^a P_{k'}^{a'} P_l^b P_{l'}^{b'} P_{j'}^{b'} P_j^b P_{i'}^{a'} P_i^a | \psi \rangle. \quad (15)$$

(This is just a decoherence functional in the sense of [19], evaluated on the coarse-grained histories represented by $P_i^a P_{i'}^{a'} P_j^b P_{j'}^{b'}$, with initial state $|\psi\rangle$.)

The decoherence functional of (15) will do the job, but there’s a nicer, more symmetric form that will also work, where the order of the \mathbf{a} and \mathbf{a}' measurements is symmetrized and similarly for \mathbf{b} and \mathbf{b}' :

$$\begin{aligned} &\widehat{D}_{sym}(ii'jj'; kk'll') \\ &= \frac{1}{16} \langle \psi | (P_k^a P_{k'}^{a'} + P_{k'}^{a'} P_k^a) (P_l^b P_{l'}^{b'} + P_{l'}^{b'} P_l^b) (P_j^b P_{j'}^{b'} + P_{j'}^{b'} P_j^b) (P_i^a P_{i'}^{a'} + P_{i'}^{a'} P_i^a) | \psi \rangle. \end{aligned} \quad (16)$$

Some simple σ -matrix algebra, using $(\boldsymbol{\sigma} + \boldsymbol{\rho})|\psi\rangle = 0$ and $\langle \psi | \boldsymbol{\sigma} | \psi \rangle = 0$, because $|\psi\rangle$ is the singlet, gives

$$\begin{aligned} &256 \widehat{D}_{sym}(ii'jj'; kk'll') \\ &= (1 + ik + i'k')(1 + jl + j'l') \\ &\quad + (ik' - i'k)(jl' - j'l) \\ &\quad - \frac{1}{\sqrt{2}}(i + k)(j + l + j' + l') \\ &\quad + \frac{1}{\sqrt{2}}(i' + k')(j + l - j' - l'). \end{aligned} \quad (17)$$

One can easily verify that $|Q| = 2\sqrt{2}$ with these numbers.

The 16×16 matrix \widehat{D}_{sym} has 12 zero eigenvalues and so the Hilbert space that one constructs from it is four-dimensional, as one would expect for a pair of spin- $\frac{1}{2}$ particles [15]. The existence of null directions also means that $\widehat{\mu}$ is verging on violating strong positivity, the matrix \widehat{D} being only positive *semi*-definite.

Realizing the joint sample space

The construction we have just employed can be made more vivid by relating it to a gedankenexperiment in which the 16 “outcomes” comprising the sample space $\widehat{\Omega}$ correspond to actual trajectories of physical particles. The expression (15) can be interpreted in terms of Stern-Gerlach devices for silver atoms (or perhaps more conveniently, in terms of photon trajectories and interferometers, *cf.* the setup in [23].) An outcome of a spin measurement then amounts to a silver atom’s emerging in either the upper or lower beam. However, suppose that we don’t “look at” the silver atom, but instead send it through a reversed magnetic field designed to recombine the two beams, as if they had never been split apart at all. We can then pass it through a *second* Stern-Gerlach analyzer which again splits the beam into two, *etc.* If we concatenate two analyzers this way in region A , and two more in region B , then we naturally partition the full history space into 16 subsets depending on which beams the silver atoms traverse in their respective analyzers. In this way the elements of $\widehat{\Omega}$ are realized as actual sets of histories that all pertain to a *single* experimental setup. It follows that *any histories formulation*^{*} *must induce in this manner a quantal measure on $\widehat{\Omega}$* ; and the matrix-element (15) that we wrote down before is just the algebraic expression of this measure.

Of course, the mere fact that the measure $\widehat{\mu}$ is well defined does not yet tell us what will happen if we do “look at” the particles. To make contact with the experimental probabilities $p(\alpha_i, \beta_j)$, we must assume further that if we do choose to look, then the measure induced thereby on “us” directly reflects the measure $\widehat{\mu}$ on the space $\widehat{\Omega}$ of “microscopic”

^{*} Any formulation, that is, for which the silver atoms are part of the kinematics (or “ontology”) and trace out continuous worldlines in spacetime. With discontinuous trajectories, the silver atoms might be present in both beams, and an event like “silver atom in upper beam in first analyzer” would not be well defined. It seems that something like this would actually occur in models such as that of [24].

alternatives. That is, we must assume that “looking” at a beam *merely reveals* the corresponding value of $\hat{\mu}$. (Note in this connection that events in distinct spatial locations at a given time always decohere in unitary quantum mechanics.)

IV. Relation to the “screening off” causality condition

Classical case

The condition of *screening off* on the classical measure μ asserts that events in causally unrelated regions A, B of spacetime become independent (“screened off” from one another) when one conditions on a complete specification c of the history in the region $C = J^-(A) \cap J^-(B)$, the mutual causal past[†] of A and B . The logic underlying this condition is that any correlation between spacelike separated variables must arise entirely from their separate correlations with some “common cause” in their mutual past, and therefore must disappear once full information about the past is given.^b

Specialized to our situation, this screening off condition yields for all $\alpha \in M_A, \beta \in M_B$ and $i, j = \pm 1$,

$$\mu(\alpha_i \cap \beta_j | c) = \mu(\alpha_i | c) \mu(\beta_j | c) , \quad (18)$$

where c is any subset of Ω defined by a completely fine grained specification of the history in $C = J^-(A) \cap J^-(B)$. Similarly (or just by summing (18) on i, j), we have

$$\mu(\alpha \cap \beta | c) = \mu(\alpha | c) \mu(\beta | c), \quad (19)$$

so that the “setting event” α is screened off from the setting event β . Dividing (18) by (19) yields an equation which, we claim, can be written as

$$\mu(\alpha_i \cap \beta_j | \alpha \cap \beta \cap c) = \mu(\alpha_i | \alpha \cap c) \mu(\beta_j | \beta \cap c) . \quad (20)$$

[†] This is a strong form of the screening off condition, as it excludes in particular “primordial correlations”. A less restrictive condition, depending on the context, would locate c in the *union* of the (exclusive) pasts of A and B . For details see [9].

^b This is not the only way to construe the “principle of common cause”, but it is the one adopted in all discussions of the Bell inequalities known to us.

This follows from noting that

$$\begin{aligned}
\mu(\alpha_i \cap \beta_j | \alpha \cap \beta \cap c) &= \frac{\mu(\alpha_i \cap \beta_j \cap \alpha \cap \beta \cap c)}{\mu(\alpha \cap \beta \cap c)} \\
&= \frac{\mu(\alpha_i \cap \beta_j \cap c)}{\mu(\alpha \cap \beta \cap c)} && [\text{since } \alpha_i \subset \alpha, \beta_j \subset \beta] \\
&= \frac{\mu(\alpha_i \cap \beta_j \cap c)/\mu(c)}{\mu(\alpha \cap \beta \cap c)/\mu(c)} \\
&= \frac{\mu(\alpha_i \cap \beta_j | c)}{\mu(\alpha \cap \beta | c)}
\end{aligned}$$

and

$$\begin{aligned}
\mu(\alpha_i | \alpha \cap c) &= \frac{\mu(\alpha_i \cap \alpha \cap c)}{\mu(\alpha \cap c)} \\
&= \frac{\mu(\alpha_i \cap c)}{\mu(\alpha \cap c)} \\
&= \frac{\mu(\alpha_i \cap c)/\mu(c)}{\mu(\alpha \cap c)/\mu(c)} \\
&= \frac{\mu(\alpha_i | c)}{\mu(\alpha | c)},
\end{aligned}$$

and similarly for $\mu(\beta_j | \beta \cap c)$. [In these calculations, one is in effect “conditioning in stages” and recognizing that $\mu((x|y)|z) = \mu(x|y \cap z)$, where $\mu((x|y)|z) := \mu(x \cap y | z) / \mu(y | z)$.] Observe that, in order for the conditional probabilities appearing in (18)–(20) to be defined, none of the measures $\mu(c)$, $\mu(\alpha \cap c)$, $\mu(\beta \cap c)$, $\mu(\alpha \cap \beta \cap c)$ can vanish. Accordingly, (20) is only valid with this reservation.

At this point, we need to formalize the idea that the instrument settings are “chosen freely”. To that end, we will assume that, with respect to the measure μ , and for all $\alpha \in M_A$ and $\beta \in M_B$, the “setting events” α and β are independent of any \star events in C . In the presence of screening off, this implies that the event “ α and β ” is also independent of any event in C :

$$\mu(\alpha \cap \beta \cap c) = \mu(\alpha \cap \beta)\mu(c) . \tag{21}$$

\star This is a rather drastic form of setting-independence. It would have been possible to include other events in the past on which the settings depended without affecting the main points of the argument, as in [10].

(It also implies that $\mu(\alpha \cap \beta) = \mu(\alpha)\mu(\beta)$, so that, in the presence of screening off, the setting events are strictly independent of one another.) The formal derivation of (21) goes as follows. Our assumption of “setting-independence” says that

$$\mu(\alpha|c) = \mu(\alpha) \quad (\text{and similarly for } \beta) . \quad (22)$$

Putting this together with (19), we obtain $\mu(\alpha \cap \beta|c) = \mu(\alpha|c)\mu(\beta|c) = \mu(\alpha)\mu(\beta)$, whence $\mu(\alpha \cap \beta \cap c) = \mu(\alpha)\mu(\beta)\mu(c)$, whence $\mu(\alpha \cap \beta) = \mu(\alpha)\mu(\beta)$ by summing on c . Comparing the first and last equations yields $\mu(\alpha \cap \beta|c) = \mu(\alpha \cap \beta)$, which is (21). Note finally that we can assume without loss of generality that $\mu(c) > 0$ for all fine-grained specifications c of C (otherwise simply omit c from Ω). Then, having just demonstrated that $\mu(\alpha \cap \beta \cap c) = \mu(\alpha)\mu(\beta)\mu(c)$, we conclude that $\mu(\alpha \cap \beta \cap c)$ never vanishes (unless we can’t do the experiment at all!); hence equation (20) becomes valid unreservedly.

It is well known that the screening off condition leads to the CHSHB inequality [10]. This follows from a result of Fine [4] according to which the existence of a joint distribution $\widehat{\mu}$ on $\widehat{\Omega}$ is equivalent[†] to screening off. More formally, let us say that a system of experimental probabilities $p(\alpha_i, \beta_j)$ admits a classical screening off model if one can find a sample space Ω and a measure μ thereon obeying (18) and (21), and such that $\mu(\alpha_i \cap \beta_j | \alpha \cap \beta) = p(\alpha_i, \beta_j)$. Then

Lemma 4.1 A system of experimental probabilities admits a classical screening off model if and only if it admits a joint probability distribution $\widehat{\mu}$.

[†] Fine’s treatment appears to rely tacitly on the “non-contextuality” assumption that settings of the remote instrument cannot affect local results. The condition that he invokes is not actually screening off as such, but what he calls “factorizability”, a condition which, as he words it, seems to be ambiguous between two formulations, the first of which (corresponding to our equation (20)) could be written in our notation as $\mu(\alpha_i \cap \beta_j | \alpha \cap \beta \cap c) = \mu(\alpha_i | \alpha \cap c)\mu(\beta_j | \beta \cap c)$, and the second of which would be $\mu(\alpha_i \cap \beta_j | \alpha \cap \beta \cap c) = \mu(\alpha_i | \alpha \cap \beta \cap c)\mu(\beta_j | \alpha \cap \beta \cap c)$. In these expressions, however, “conditioning” on α (for example) merely means that the instrument at A is set to α . Fine avoids attributing probabilities to instrument settings, in contrast to the approach we have adopted in this paper; his is the “minimalist approach” mooted at the beginning of Section III.

PROOF (1) Let (Ω, μ) be a screening off model for some system of experimental probabilities. We must demonstrate that there exists a classical measure, $\hat{\mu}$ on $\hat{\Omega}$, such that

$$\mu(a_i \cap b_j | a \cap b) = \sum_{i'j'} \hat{\mu}(i' j' j')$$

and similarly for each (α, β) pair. In the following, recall that c ranges over all subsets of Ω specified by a complete fine-grained description of $J^-(A) \cap J^-(B)$ for which $\mu(c) > 0$. Recall also that we have assumed that Ω has finite cardinality.

Since, by our assumptions, $\mu(a \cap c)$ never vanishes, $\mu(a_i | a \cap c)$ is defined, and we have

$$\sum_i \mu(a_i | a \cap c) = 1, \quad (23)$$

and similarly for a', b, b' . Now make the “maximal independence ansatz”,

$$\hat{\mu}(i' j' j') = \sum_c \mu(a_i | a \cap c) \mu(a'_{i'} | a' \cap c) \mu(b_j | b \cap c) \mu(b'_{j'} | b' \cap c) \mu(c). \quad (24)$$

We claim that the marginals of $\hat{\mu}$ agree with μ for each (α, β) pair. For example, take $\alpha = a, \beta = b$; then

$$\begin{aligned} & \sum_{i'j'} \hat{\mu}(i' j' j') \\ &= \sum_c \mu(a_i | a \cap c) \mu(b_j | b \cap c) \mu(c) \quad [\text{by (23)}] \\ &= \sum_c \mu(a_i \cap b_j | a \cap b \cap c) \mu(c) \quad [\text{by (20)}] \\ &= \sum_c \frac{\mu(a_i \cap b_j \cap c)}{\mu(a \cap b \cap c)} \mu(c) \\ &= \sum_c \frac{\mu(a_i \cap b_j \cap c)}{\mu(a \cap b) \mu(c)} \mu(c) \quad [\text{by (21)}] \\ &= \frac{\sum_c \mu(a_i \cap b_j \cap c)}{\mu(a \cap b)} \\ &= \frac{\mu(a_i \cap b_j)}{\mu(a \cap b)} \\ &= \mu(a_i \cap b_j | a \cap b). \end{aligned}$$

(2) Conversely, suppose there exists a classical measure $\widehat{\mu}$ on $\widehat{\Omega}$ as described above with marginals that agree with μ on each (α, β) pair. A consistent screening off model can be found by supposing that there were other events in the past which were not taken into account in the original sample space Ω .

We will assume for the purposes of this proof that the original sample space Ω contains nothing but the experimental events of interest, *i.e.* the settings and outcomes, and in particular contains no events in the mutual past of A and B . It would have been possible to include past events of which all the experimental probabilities in Ω were independent; no essentially new idea is needed to extend the proofs to this case, but they are excluded here for the sake of clarity.

Let $\widetilde{\Omega}$ be a new sample space whose fine-grained histories are those of Ω with an additional quadruple of binary variables which we will regard as residing in the mutual past of A and B . (These variables play the role of the past “causes” of the experimental outcomes.) We claim that there exists a classical measure $\widetilde{\mu}$ on $\widetilde{\Omega}$ such that the measure $\widetilde{\mu}$ agrees with μ on all of the experimental events — and that $\widetilde{\mu}$ satisfies screening off. To demonstrate this, let us set, formally,

$$\widetilde{\Omega} = \{(h, i, i', j, j') \mid h \in \Omega \text{ and } i, i', j, j' \in \{+1, -1\}\}$$

and declare by fiat that the quadruple of binary variables (i, i', j, j') lives in the mutual past of A and B . Then any subset of Ω can be considered a subset of $\widetilde{\Omega}$ in the obvious way. We write $\{ii'jj'\}$ or $\{kk'll'\}$ for the set of all histories in $\widetilde{\Omega}$ with those particular values of the quadruple in the past.

The statement of screening off for this new model is

$$\widetilde{\mu}(\alpha_i \cap \beta_j \mid \{kk'll'\}) = \widetilde{\mu}(\alpha_i \mid \{kk'll'\})\widetilde{\mu}(\beta_j \mid \{kk'll'\}). \quad (25)$$

We must find a $\widetilde{\mu}$ which extends μ and for which (25) holds. Note that the experimental settings are still required to be independent of all past events, which now means $\{kk'll'\}$.

We make the ansatz

$$\begin{aligned} \widetilde{\mu}(a_i \cap b_j \cap \{kk'll'\}) &= \mu(a \cap b) \widehat{\mu}(kk'll') \delta_{ik} \delta_{jl} \\ \widetilde{\mu}(a'_{i'} \cap b_j \cap \{kk'll'\}) &= \mu(a' \cap b) \widehat{\mu}(kk'll') \delta_{i'k'} \delta_{jl} \end{aligned} \quad (26)$$

and similarly for the other two (α, β) pairs, (a, b') and (a', b') .

Summing one example of (26) over k, k', l, l' yields, with the help of (3),

$$\tilde{\mu}(a_i \cap b_j) = \mu(a \cap b) \sum_{i'j'} \hat{\mu}(ii'jj') = \mu(a_i \cap b_j) .$$

This shows that the probabilities of the experimental outcomes and settings are the same for μ and $\tilde{\mu}$.

As required, the settings are also independent of the added past variables with respect to $\tilde{\mu}$, for example:

$$\begin{aligned} \tilde{\mu}(a \cap b \cap \{kk'll'\}) &= \sum_{ij} \tilde{\mu}(a_i \cap b_j \cap \{kk'll'\}) \\ &= \mu(a \cap b) \hat{\mu}(kk'll') \\ &= \tilde{\mu}(a \cap b) \tilde{\mu}(\{kk'll'\}) . \end{aligned}$$

The ansatz (26) also gives, after simple manipulations,

$$\begin{aligned} \tilde{\mu}(a_i \cap b_j \mid \{kk'll'\}) &= \delta_{ik} \delta_{jl} \tilde{\mu}(a \cap b) \\ \tilde{\mu}(a_i \mid \{kk'll'\}) &= \delta_{ik} \tilde{\mu}(a) \\ \tilde{\mu}(b_j \mid \{kk'll'\}) &= \delta_{jl} \tilde{\mu}(b) , \end{aligned}$$

which implies (25), so the new measure $\tilde{\mu}$ satisfies screening off. QED

The significance of the second part of the lemma is that, given the existence of the joint probability measure $\hat{\mu}$ on $\hat{\Omega}$, the observed experimental probabilities can always be explained classically and causally, in a suitably chosen model. (In the proof of this part of the lemma, the underlying idea is almost trivial, despite the somewhat complicated notation that expresses it in this case: if the past *determines* the future, then *any* two future events become independent when the past is conditioned upon. Screening off is thus automatic in any “deterministic” situation. The same basic fact persists for quantal measures and will underlie our proof of Lemma 4.2 in the next subsection (where the notational complications are even greater).)

Corollary Screening off \Rightarrow the CHSHB inequality.

REMARK By conditioning on a given instrumental setup (α, β) , we obtain from the overall measure μ a probability measure on the space $W_{\alpha\beta} = \Omega_\alpha \times \Omega_\beta \times \Omega_C$, where Ω_C is the space of all configurations or “partial histories” in the past region C . In this way, we obtain four distinct “measure spaces”, where by this phrase we simply mean a sample-space endowed with a measure. The proof of part (1) of the lemma in effect “patches” these four measure spaces together into a single measure space $W = \Omega_a \times \Omega_{a'} \times \Omega_b \times \Omega_{b'} \times \Omega_C$, in a manner reminiscent of a fibre product, with Ω_C playing the role of common base space. The ansatz (24) then produces $\widehat{\Omega}$ (with $\widehat{\mu}$) as the marginal measure space resulting from W by neglecting C .

An alternative to screening off?

This might be an appropriate place to comment on the possibility of a different derivation of the CHSHB inequality, in which an enhanced locality condition does some of the work done by screening off in the proof of Lemma 4.1. Can one, in fact, demonstrate the existence of a joint decoherence functional without invoking screening off as such? This is an interesting question because it is perhaps not settled that screening off is the true expression of relativistic causality, even in the classical case [9] [25]. Here, then, is such an alternative derivation (albeit not as precisely formulated).

We start from the assumption that instruments at A (resp. B) respond only to certain local variables (“beables”) ξ_A (resp. ξ_B) defined in A (resp. B). The perfect correlations that arise in the singlet state then imply that these local variables *determine* the response unambiguously, without any stochastic component (this being the EPR observation); and we may assume that this is always so, even in examples such as that of [6], where the correlations are not perfect. On this basis, we immediately acquire our sample space $\widehat{\Omega}$, parameterized by the values of the local variables $\xi_{A,B}$.

We also need a matching probability measure $\widehat{\mu}$ on $\widehat{\Omega}$. For this, we must assume that the choice of instrument setting at A — including the choice of no measurement at all — cannot influence local variables at B , and vice versa for settings at B . It follows that we get a well defined (setting independent) probability distribution on the variables ξ_A, ξ_B , and this induces a probability measure on $\widehat{\Omega}$, whose marginals are obviously the experimental probabilities, $p(a_i, b_j)$, *etc.*

We see that screening off as such was not used. In its place was the assumption that instrument settings do not influence the “hidden variables” ξ_A and ξ_B . We tacitly assumed as well, of course, that the variables ξ do not influence the instrument settings, *i.e.* that the latter were “free” in relation to this particular set of microscopic variables. Notice also that the derivation in this form did not require us to attribute probabilities to instrument settings, except insofar as this would be one way to make precise their “freedom” in respect of the variables ξ .

In the derivation just described, the “local beables” ξ provide a “material basis” for the sample space $\widehat{\Omega}$, in the sense that elements of $\widehat{\Omega}$ represent equivalence classes of histories determined by the values of those beables. In contrast, the proof we gave earlier merely *concocted* a space $\widehat{\Omega}$ (and a measure on it), without attempting to identify it with any actual set of physical histories (*cf.* the remarks under “Realizing the joint sample space”, above.)

Quantal case; Proposal for quantal screening off

We have seen that classically, there is an equivalence between screening off and the existence of a joint probability measure $\widehat{\mu}$ on all of the outcomes under consideration, *i.e.* on $\widehat{\Omega}$. We would like to prove something similar in the quantum case: that a suitably generalized screening off condition is equivalent to the existence of a joint *quantal* measure on $\widehat{\Omega}$, all subject to appropriate conditions of setting independence and decoherence. Given our standing assumption of strong positivity, the Tsirel’son inequality would then follow from quantal screening off.

We will propose a candidate for a condition of quantum screening off such that, if there exists a joint decoherence functional \widehat{D} with the correct marginals, then a past can be cooked up, just as in the classical case, so that the resulting quantum measure $\widetilde{\mu}$ satisfies the proposed condition. The converse of this, that the candidate quantum screening off condition implies the existence of a strongly positive joint decoherence functional on $\widehat{\Omega}$ with the correct marginals remains conjectural for now. We will, however, prove that such a decoherence functional on $\widehat{\Omega}$ exists in the case of ordinary unitary quantum mechanics, and we will highlight the causality assumption that allows the construction.

To motivate the proposed quantal screening off condition, notice that the *classical* screening off condition (18) is equivalent to

$$\mu(\alpha_i \cap \beta_j \cap c)\mu(c) = \mu(\alpha_i \cap c)\mu(\beta_j \cap c) .$$

The analogous condition on D is then our proposal for quantum screening off:

$$D(\alpha_i \cap \beta_j \cap c; \bar{\alpha}_k \cap \bar{\beta}_l \cap \bar{c})D(c; \bar{c}) = D(\alpha_i \cap c; \bar{\alpha}_k \cap \bar{c})D(\beta_j \cap c; \bar{\beta}_l \cap \bar{c}) , \quad (27)$$

for all settings $\alpha, \bar{\alpha}$ and $\beta, \bar{\beta}$, and for all fully specified pasts c, \bar{c} [as in (18)]. More details and a proof that quantum field theory satisfies this condition formally will appear in a separate work [25]. (Conditions (27) include matrix elements that are off-diagonal in the instrument settings, α and β . With the “minimalist approach”, only the equalities with $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$ would be meaningful.) Notice that if D is completely diagonal, then (27) reduces to classical screening off. A variation on (27) asserts (in a shorthand notation) that ^b

$$D(ijc; \bar{k}\bar{l}\bar{c}) D(pqc; \bar{r}\bar{s}\bar{c}) = D(pjc; \bar{r}\bar{l}\bar{c}) D(iqc; \bar{k}\bar{s}\bar{c}) \quad (28)$$

where c and \bar{c} are as before, every other index stands for an event in region A or B , and indices appear in the order: A -event, B -event, mutual past. From (28) one can deduce that D decomposes as a product of the form

$$D(ijc; \bar{k}\bar{l}\bar{c}) = F(ic; \bar{k}\bar{c}) G(jc; \bar{l}\bar{c}) . \quad (29)$$

This formulation carries more information than (27) when $D(c; \bar{c}) = 0$, which can happen non-trivially in the quantal case.

We also take the quantum condition of setting independence to be

$$\begin{aligned} D(\alpha \cap \beta \cap c; \alpha \cap \beta \cap \bar{c}) &= D(\alpha; \alpha)D(\beta; \beta)D(c; \bar{c}) \\ &= \mu(\alpha)\mu(\beta)D(c; \bar{c}), \end{aligned}$$

where $\alpha \in M_A, \beta \in M_B; c$ and \bar{c} can be any two events in C ; and we have assumed that D is diagonal in α and β . Finally, recall that all decoherence functionals are assumed by default to be strongly positive.

^b The pattern might clarify as: $D(ijk; lmn)D(i'j'k; l'm'n) = D(i'jk; l'mn)D(ij'k; lm'n)$

In the next lemma, the augmented history space $\tilde{\Omega}$ is the same space as appeared in the proof of Lemma 4.1, part 2. Also, of course, μ is the quantal measure on Ω and D its associated decoherence functional.

Lemma 4.2 Let D be a decoherence functional on Ω that decoheres on (α, β) pairs and such that α and β are independent of everything else. Assume that the induced experimental probabilities $p(\alpha_i, \beta_j)$ admit a joint quantal measure in the sense of the definition given above in equations (7) and (8). Then there exists a (strongly positive) decoherence functional \tilde{D} on $\tilde{\Omega}$ that agrees with D on all pairs of instrument events, and that satisfies quantum screening off.

PROOF As in the classical case, we assume that Ω contains no “irrelevant” events. We again concoct extra events $\{ii'jj'\}$ in the region C that were not taken into account in Ω . Our screening off condition for the new model based on $\tilde{\Omega}$ is

$$\begin{aligned} & \tilde{D}(\alpha_i \cap \beta_j \cap \{kk'll'\}; \bar{\alpha}_m \cap \bar{\beta}_n \cap \{pp'qq'\}) \tilde{D}(\{kk'll'\}; \{pp'qq'\}) \\ &= \tilde{D}(\alpha_i \cap \{kk'll'\}; \bar{\alpha}_m \cap \{pp'qq'\}) \tilde{D}(\beta_j \cap \{kk'll'\}; \bar{\beta}_n \cap \{pp'qq'\}) , \end{aligned} \quad (30)$$

$\forall \alpha$ and β . Define the decoherence functional \tilde{D} on $\tilde{\Omega}$ by the equations,

$$\begin{aligned} & \tilde{D}(a_i \cap b_j \cap \{kk'll'\}; a_m \cap b_n \cap \{pp'qq'\}) \\ &= \delta_{ik} \delta_{jl} \delta_{mp} \delta_{nq} \mu(a \cap b) \hat{D}(kk'll'; pp'qq') , \end{aligned} \quad (31)$$

and similarly for the 3 other (α, β) pairs, taking \tilde{D} to vanish when the instrument settings are off-diagonal *e.g.*

$$\tilde{D}(a'_i \cap b_j \cap \{kk'll'\}; a_m \cap b_n \cap \{pp'qq'\}) = 0 .$$

From this, it can be seen that $\tilde{D}(\{kk'll'\}; \{pp'qq'\}) = \hat{D}(kk'll'; pp'qq')$.

Now, summing, for example, (31) over k, k', l, l' and p, p', q, q' produces

$$\begin{aligned} \tilde{D}(a_i \cap b_j; a_m \cap b_n) &= \mu(a \cap b) \sum_{k'l'p'q'} \hat{D}(ik'jl'; mp'nq') \\ &= \mu(a \cap b) p(a_i, b_j) \delta_{im} \delta_{jn} \\ &= \mu(a_i \cap b_j) \delta_{im} \delta_{jn} \\ &= D(a_i \cap b_j; a_m \cap b_n) \end{aligned}$$

using (8), (7) and (6). This shows that \tilde{D} takes the same values as D for all pairs of experimental settings and outcomes.

As required, the setting-events are also independent of the added past variables with respect to \tilde{D} , for example:

$$\begin{aligned} \tilde{D}(a \cap b \cap \{kk' ll'\}; a \cap b \cap \{pp' qq'\}) &= \sum_{ijmn} \tilde{D}(a_i \cap b_j \cap \{kk' ll'\}; a_m \cap b_n \cap \{pp' qq'\}) \\ &= \mu(a \cap b) \hat{D}(kk' ll'; pp' qq') \\ &= \tilde{\mu}(a \cap b) \tilde{D}(\{kk' ll'\}; \{pp' qq'\}) . \end{aligned}$$

The definition of \tilde{D} also gives

$$\begin{aligned} \tilde{D}(a_i \cap \{kk' ll'\}; a_m \cap \{pp' qq'\}) &= \delta_{ik} \delta_{mp} \tilde{\mu}(a) \hat{D}(kk' ll'; pp' qq') \\ \tilde{D}(b_j \cap \{kk' ll'\}; b_n \cap \{pp' qq'\}) &= \delta_{jl} \delta_{nq} \tilde{\mu}(b) \hat{D}(kk' ll'; pp' qq') , \end{aligned}$$

which implies (30) for $\alpha = \bar{\alpha} = a$ and $\beta = \bar{\beta} = b$. Similar calculations can be done for every (α, β) pair, and when the instrument settings are off-diagonal (30) holds trivially as both sides are zero. The new measure $\tilde{\mu}$ thus satisfies quantum screening off.

Finally, \tilde{D} is strongly positive because it is essentially just \hat{D} which is strongly positive by assumption.

QED

We lack a proof of the converse of Lemma 4.2 (an analogue of part 1 of Lemma 4.1). But we *can* show that a strongly positive decoherence functional on $\hat{\Omega}$ with the correct marginals, and which decoheres on all (α, β) pairs, exists in the case of unitary quantum mechanics (*cf.* our earlier discussion of concatenated Stern-Gerlach beam splitters with “recombiners”, which suggests more generally that $\hat{\Omega}$ should be realizable in any “histories formulation”).

In standard quantum mechanics, for any measurement α in A , there exist projection operators P_i^α , $i = \pm 1$, which project onto the subspaces of Hilbert space associated with the outcomes ± 1 of the measurement. Plainly, $P_{+1}^\alpha + P_{-1}^\alpha = 1$. Similarly there exist operators P_j^β , $j = \pm 1$, projecting onto the subspaces of Hilbert space associated with

the outcomes ± 1 of the measurement β in B . The standard causality assumption is then $[P_i^\alpha, P_j^\beta] = 0$.

Given this, it is easy to construct, analogously to (15), a joint decoherence functional on $\widehat{\Omega}$ with the desired properties. Let

$$\widehat{D}(ii'jj'; kk'll') = \text{Tr}(P_{j'}^{b'} P_j^b P_{i'}^{a'} P_i^a \rho_0 P_k^a P_{k'}^{a'} P_l^b P_{l'}^{b'}), \quad (32)$$

where ρ_0 is the density matrix giving the pre-measurement state of the particles and the trace is over particle states.

Lemma 4.3 \widehat{D} is strongly positive and has the correct marginals (8).

PROOF \widehat{D} has the canonical form of a decoherence functional of ordinary unitary quantum mechanics, which is known to be strongly positive [21]. To show that it has the correct marginals (8) on each (α, β) pair, let us work out for example the case $(\alpha, \beta) = (a, b)$:

$$\begin{aligned} \sum_{i'j'k'l'} \widehat{D}(ii'jj'; kk'll') &= \text{Tr}(P_j^b P_i^a \rho_0 P_k^a P_l^b) \\ &= \text{Tr}(P_i^a \rho_0 P_k^a P_j^b) \delta_{jl} \\ &= \text{Tr}(\rho_0 P_i^a P_j^b) \delta_{ik} \delta_{jl} = p(a_i, b_j) \delta_{ik} \delta_{jl}, \end{aligned}$$

using the cyclic property of the trace and $P_l^b P_j^b = P_j^b \delta_{jl}$ in the middle line, and the posited commutativity of P_i^a and P_j^b in the last. QED

Notice that \widehat{D} is only one of many decoherence functionals which satisfy the desired conditions. Instead of the product $P_i^a P_{i'}^{a'}$, for example, we could have any convex combination of $P_i^a P_{i'}^{a'}$ and $P_{i'}^{a'} P_i^a$. (It seems unlikely that the most general \widehat{D} can be obtained in this manner, though, because our ansatz here exhibits an extra decoherence not demanded by the physics; for example, $\widehat{D}(ij; k'l) := \sum_{i'j'k'l'} \widehat{D}(ii'jj'; kk'll') \propto \delta_{jl}$ because $\text{Tr}(P_j^b P_i^a \rho_0 P_{k'}^{a'} P_l^b) \propto P_j^b P_l^b \propto \delta_{jl}$, even though $a \neq a'$.)

REMARK Even without commutativity, the above trace expressions would define decoherence functionals \widehat{D} and $\widehat{D}_{\alpha\beta}$ for $\widehat{\Omega}$ and the $\widehat{\Omega}_{\alpha\beta}$, and the marginals of \widehat{D} would still reproduce the $\widehat{D}_{\alpha\beta}$. In light of this, one might perceive the existence of a joint quantal measure as reflecting most directly the existence for the $\widehat{D}_{\alpha\beta}$ of trace expressions involving operators in a common Hilbert space (cf. [26]). The commutativity would manifest itself,

on this view, only in the fact that distinct α -outcomes continue to decohere independently of whether a β -measurement is made, and independently of its outcome if it is.

In the classical context, the existence of a joint probability distribution $\hat{\mu}$ on $\hat{\Omega}$ is often described by saying that one can find non-contextual hidden variables capable of reproducing the given system of experimental probabilities. Adopting the same language, we can interpret Lemma 4.3 in the following manner: It is possible to attribute the correlations in the EPRB setup to non-contextual* hidden variables, so long as they are *quantal* hidden variables, governed by a decoherence functional rather than a classical probability distribution.†

V. Weak positivity is not enough

We have seen that the condition of strong positivity leads to the Tsirel'son inequality. Can the inequality be violated by a decoherence functional that is only weakly positive (but otherwise observes the conditions of theorem 2)? That this is so can be seen simply by noting the continuity of Q in equation (13), and checking that \hat{D}_{sym} in equation (17) is not on the boundary of *weak* positivity — meaning that it assigns no set a measure of exactly zero. We have verified this for all of the $2^{16} - 1$ non-empty subsets of Ω .

* By “non-contextual” we refer to the fact that the quantal measure $\hat{\mu}$ on $\hat{\Omega}$ is defined *independently of any measuring instruments or their settings*. In this sense, one can say that a given measurement (if suitably designed) “merely reveals” a particular value of $\hat{\mu}$, without participating in its definition. (In saying this, we are not asserting that, in any individual instance, the measurement “merely reveals”, for example, the location of the silver atom without affecting it. This would be a much stronger claim, and possibly meaningless in a non-deterministic theory which provides no account of “what would have happened” in any individual instance, had the measurement not taken place.)

† That a non-contextual quantal measure $\hat{\mu}$ exists where a non-contextual classical measure cannot, implies that the corresponding decoherence functional \hat{D} fails to be diagonal; for a decoherence functional is classical if it is diagonal. And indeed the \hat{D} constructed above in the case of unitary quantum theory is easily seen to possess off-diagonal matrix elements. In the framework of the “consistent histories” point of view this means that the coarse-grained histories specified by $(a_i, a'_{i'}, b_j, b'_{j'})$ fail to decohere, and it is consequently not possible to assign probabilities simultaneously to all of the “quantal hidden variables”.

In fact, one might go further and ask whether the maximum possible value of $Q = 4$ can be attained by a weakly positive decoherence functional. The answer is yes, and there exist remarkably simple examples. Here are the elements of one such example obtained by the `lp_solve` linear programming solver [27].

$$\begin{aligned}
\widehat{D}(- - - -; - - - -) &= \widehat{D}(+ - +-; + - +-) = \widehat{D}(+ - - +; + - - +) \\
&= \widehat{D}(- + - +; - + - +) = \frac{1}{2} \\
-\widehat{D}(- - - +; - - - -) &= -\widehat{D}(+ - +-; + - - -) = \widehat{D}(- + - +; - + - -) \\
&= \widehat{D}(- - - +; + + - -) = -\widehat{D}(+ - - +; + + - -) \\
&= -\widehat{D}(- + - +; + + - -) = \widehat{D}(+ - - +; + - +-) \\
&= -\widehat{D}(+ - - +; - - - +) = -\widehat{D}(- + - +; - - - +) \\
&= \widehat{D}(- + - +; + - - +) = \frac{1}{4}
\end{aligned}$$

The remaining elements which are not equal to one of the above by Hermiticity are zero. For this decoherence functional, one checks that

$$X(a, b) - X(a', b) + X(a, b') + X(a', b') = 4$$

For consistency with theorem 2, strong positivity must be violated by any \widehat{D} which violates the Tsirel'son bound. One can check that the above \widehat{D} does so, with four negative signs, four positive signs, and eight zeros in its signature.

In the context of the Bell inequalities, then, the strong positivity condition of quantum measure theory shows itself to be much stronger than the weak one. To the extent that weak positivity is physically acceptable, one can imagine a generalized form of quantum mechanics (a generalized measure theory remaining at level two) which affords the maximum possible violation of the CHSHB inequality. Strong positivity, in contrast, is as restrictive as ordinary quantum mechanics in this respect.

One other feature of the above matrix \widehat{D} seems worthy of notice here. All the marginals of the form $\widehat{\mu}(a_{\pm})$, $\widehat{\mu}(a'_{\pm})$, *etc.* take the value $1/2$, which is recognizable as the only ‘causal’ value. That \widehat{D} yields $Q = 4$ implies perfect correlations (or anti-correlations) between A and B , and any other marginals than $1/2$ would let Alya signal to Bai by manipulating

the settings of her analyzer. But this could not happen with the above \widehat{D} because “non-signaling” is built into the requirements we have imposed on it in equations (8), which imply directly that $\sum_j p(a_i, b_j) = \sum_{j'} p(a_i, b'_{j'})$.

VI. Conclusion

One can view quantum mechanics as a dynamical schema that generalizes the classical theory of stochastic processes in such a manner as to take into account interference between pairs of alternatives. Within the framework appropriate to such a view — that of “quantal measure theory” — we have sought quantal analogs of some of the relationships that emerge in connection with correlated pairs of spin- $\frac{1}{2}$ particles when one contemplates tracing their behavior to the dynamics of some underlying stochastic, but still classical, variables (“hidden variables”). One knows that classically, the existence of a joint probability measure on the space of experimental outcomes is equivalent on one hand to the CHSHB inequality, and on the other hand to screening off. (This equivalence shows that the existence of hidden variables is intimately linked to causality.) Quantally, one might desire an analogous set of equivalences relating (1) the existence of a joint decoherence functional on the space $\widehat{\Omega}$ of experimental outcomes; (2) Tsirel’son’s inequality; and (3) some quantal causality condition generalizing classical screening off. We have shown — assuming strong positivity of the decoherence functional — that (1) implies (2), and that (1) also implies (3) if the latter is represented by the candidate condition (27). A proof of the converse, that (3) implies (1), would greatly strengthen the links with causality. We did not provide such a proof in general, but we did show that (1) follows from standard, unitary quantum mechanics with spacelike commutativity.

It is perhaps worth emphasizing that, just as the CHSHB inequality follows from the exceedingly general assumption of the existence of a joint probability distribution on $\widehat{\Omega}$ (in effect, a probability distribution for non-contextual hidden variables), making no statements concerning the nature of the classical dynamics save that it is given by a probability measure on a suitable history-space, so also the Tsirel’son inequality is a consequence only of the bilinear (level 2) structure of quantum theory. We have seen in fact that it follows from the mere existence of a (strongly positive) joint decoherence functional, without making any assumption that the latter has the form taken by ordinary unitary quantum mechanics. The inequality, is in this sense a statement concerning the

predictive structure of quantum mechanics itself, rather than anything to do with any specific dynamical law.

If strong positivity is discarded, we can violate the Tsirel'son inequality with a quantal (*i.e.* level two) measure, and we have even seen that the “logical” bound of 4 for the quantity Q of equation (13) can be achieved then. The corresponding non-local correlations are of interest in information theory, since they would allow certain communication tasks to be performed with fewer classical bits transferred than are demanded in standard quantum mechanics [28]. Quantal measure theory, or equivalently generalised quantum mechanics, provides for such correlations, but only if strong positivity is relaxed to weak positivity. Whether this is physically appropriate is doubtful, however. Apart from the Hilbert space constructions that it affords, a compelling physical motivation for strong positivity concerns the composition of non-interacting sub-systems [29] [15]: Strong positivity is preserved under such composition whereas weak positivity is not.^b (Might this difference lead to experimental tests that could distinguish between the two types of positivity?)

For higher level measures, we speculate that imposing an analog of strong positivity would lead to higher level inequalities still weaker than (12), but it is beyond our current powers to pursue this idea since, beyond level two, we lack the analog of the decoherence functional, in terms of which an extension of the strong positivity condition could be framed.

Let us accept provisionally that the existence of a joint decoherence functional is a necessary condition for relativistic causality. Then we can claim the following: If an EPRB-type experimental setup is ever found to violate the Tsirel'son inequality, then all causal theories in the framework of generalised quantum mechanics with a strongly positive decoherence functional are contradicted — including ordinary unitary quantum theory. However, as long as no superluminal signaling is seen, such an experimental result would not rule out causal generalised quantum theories altogether, if one were willing to accept that the world may be described by decoherence functionals that are not strongly positive. Another alternative would be to generalize to a higher order measure, in which

^b This fact is closely related to the fact that tensor products of so called completely positive maps are also completely positive.

case the challenge would be to develop good dynamical models within this at present loosely constrained class of theories.

A convincing quantal analog of the screening off condition would have an interest going far beyond its relevance to experiments of the EPRB type. In connection with quantum gravity, the condition of “Bell causality” was the guide that led to the family of (classical) dynamical laws derived in [30] for causal sets. Screening off as such lacks a clear meaning against the backdrop of a dynamical causal structure, but Bell causality is perhaps as close as one could have come to it in the causal set context. For this reason, among others, it seems clear that progress in identifying the correct quantal analog of classical screening off would help point the way to a causality principle suitable for the needs of quantum gravity.

The existence of a joint probability measure for our 16-element sample space $\widehat{\Omega}$ can be interpreted as the necessary and sufficient condition for the existence of “hidden variables” which determine “non-contextually” the measurement outcomes. That Bell’s inequality is violated in nature tells us that no such hidden variables are possible *classically*. Not so in quantal measure theory, however, and we described a model in which the “quantal hidden variables” could be identified concretely with particle worldlines. Our main finished result in this paper was that the existence of such variables can be seen as the reason for the Tsirel’son inequalities. However, non-contextuality is only part of the story. Whether such variables can be “causal” as well as “non-contextual” is a question whose answer awaits a better understanding of the concept of “quantal screening off”.

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