SU–GP–01/06–01

An example relevant to the Kretschmann-Einstein debate

Rafael D. Sorkin

Department of Physics, Syracuse University, Syracuse, NY 13244-1130, U.S.A.
and
Queen Mary College, University of London, Mile End Road, London E1 4NS
internet address: sorkin@physics.syr.edu

Abstract

We cast the flat space theory of a scalar field in generally covariant form by introducing an auxiliary field λ. The resulting theory is couched in terms of an action integral S, and all the fields (the scalar, the spacetime metric, and λ) are dynamical in the sense of being varied freely in S. Conservation of energy-momentum emerges as a formal consequence of diffeomorphism invariance, in close analogy with the situation in ordinary general relativity.

Is it possible to lend the theory of a (classical) scalar field propagating in Minkowski spacetime a diffeomorphism-invariant formulation a la Kretschmann, and if so is this formulation in some sense artificial? If we define a theory by its field equations then the answer to the first part of our question clearly is "yes".

Let us take a real, free scalar field for illustration. Its equation of motion in flat spacetime is simply

$$\Box\phi = 0 \tag{1}$$

where $\Box = \eta^{ab}\nabla_a\nabla_b$ is the wave operator or "d'Alembertian" and η_{ab} is the Minkowski metric. To reformulate this equation covariantly, we need only replace η_{ab} by a general Lorentzian metric g_{ab} and adjoin to (1) a condition stating that g_{ab} is flat:

$$R_{abcd} = 0 \ , \tag{2}$$

R_{abcd} being the Riemann tensor of g_{ab}.

These days, however, one might want to require that a theory be more than just a set of field equations. One might want it to admit a variational formulation, and one might then expect that a "genuine" diffeomorphism invariance would have to entail, in the manner of E. Noether, identities leading to conservation of energy-momentum. The purpose of this note is merely to point out that there exists a "Kretschmannian" formulation of flat space scalar field theory enjoying all of these attributes.

To obtain (1) and (2) from a variational principle, it seems simplest just to introduce an auxiliary tensor field λ^{abcd} with the same symmetries as the Riemann tensor R_{abcd}, and then to write the action integral S in a such a way that λ^{abcd} plays the role of a Lagrange multiplier: *

$$S = -\int dV \, \frac{1}{2} \, g^{ab} \nabla_a \phi \nabla_b \phi \; + \int dV (1/4) \lambda^{abcd} R_{abcd} \tag{3}$$

In this formulation ϕ, g and λ are all treated as dynamical variables, and accordingly are to be varied freely in S. The only background structure is therefore the spacetime manifold itself, exactly as in general relativity. (To specify the theory completely, one should add topological and boundary conditions, requiring, say, that the manifold be $\mathbb{R}^4$ and that the metric be geodesically complete.)

Let us see what field equations result from our choice of S. The variation of λ^{abcd} immediately yields spacetime flatness in the form of equation (2), which of course is what it was designed to do. The variation of ϕ goes exactly as in Minkowski space, and yields the same equation of motion (1), except that η^{ab} gets replaced by g^{ab} in the definition of the operator $\Box$. However, this operator immediately reduces back to the flat space d'Alembertian (with respect to $\eta_{ab} = g_{ab}$) once one takes into account the result (2) of the λ variation.

The only real novelty arises when one varies the spacetime metric g to obtain the "Einstein equation" of this theory. In discussing this variation, I will refer to the first term in (3) as the "matter term" S_m and the second as the "gravity term" S_g. By definition, variation of the metric in S_m yields the stress-energy tensor of the scalar field:

$$\delta S_m = \int dV \, \frac{1}{2} \, T^{ab} \, \delta g_{ab} \tag{4}$$

* There is no loss of generality in restricting λ^{abcd} to the same symmetry type as R_{abcd} because any piece of λ^{abcd} with a different symmetry (belonging to a distinct Young tableau) would automatically drop out of the contraction $\lambda^{abcd} R_{abcd}$. On the other hand, taking λ^{abcd} to be a tensor is perhaps less natural than taking it to be a density of weight 1, but in the presence of a nondegenerate metric, the distinction is only one of style.

where $T^{ab} = \nabla^a\phi\nabla^b\phi - \frac{1}{2}(\nabla\phi)^2 g^{ab}$. Variation of g_{ab} in the "gravity term" then has an interesting consequence. In ordinary general relativity it would have produced the Einstein tensor G^{ab}, but here we get instead a double divergence of the Lagrange multiplier field λ^{abcd} (see the Appendix). Combining, then, δS_m with δS_g, we obtain our third and last field equation:

$$\nabla_m\nabla_n\lambda^{ambn} = T^{ab} \ , \tag{5}$$

which says that T^{ab} acts as a "source" for λ^{abcd}.

Requiring that $\delta S = 0$ has thus reproduced precisely the structure of the earlier formulation (1) (2), except that an extra equation (5) has appeared, corresponding to the presence of the extra variable λ^{abcd}. This extra equation, however is not entirely trivial. Rather it yields very directly the conservation of T^{ab}, much as the Maxwell equations yield conservation of charge, or (in even closer analogy) as the (usual) Einstein equation yields T^{ab}-conservation via the contracted Bianchi identities. In the present case, one has instead of the latter, the identity (for vanishing Riemann curvature)

$$\nabla_b\left(\nabla_m\nabla_n\lambda^{ambn}\right) \equiv 0 \ , \tag{6}$$

which follows straightforwardly from the symmetry of λ^{abcd}. In light of the field equations (5) and (2), this identity immediately implies the conservation law,

$$\nabla_b T^{ab} = 0 \ , \tag{7}$$

which we therefore obtain "twice over" as a consequence of diffeomorphism invariance, exactly as in ordinary general relativity.

This is the end of the story, except for one thing. One might worry that, besides yielding (7), our extra equation (5) could also place artificial restrictions on ϕ, beyond those implied by the Klein-Gordon equation (1). But it is easy to see that this is not the case, because (at least in flat spacetime) every conserved symmetric tensor can be expressed as the double divergence of a tensor with the symmetry type of the Riemann tensor. (A proof is sketched in the the Appendix.) Notice, however, that λ^{abcd} is not uniquely determined thereby, and our theory has in this sense a further "gauge invariance", probably traceable to the (uncontracted) Bianchi identities for R_{abcd}.

From all this, one can conclude, I think, that the Kretschmann version of flat spacetime scalar field theory exhibited above not only yields the desired equations of motion, but also enjoys all the formal features we normally associate with generally covariant theories. * In this way, far from being "artificial", it affords (just as general relativity does) a particularly simple derivation of the stress-energy tensor T^{ab} and a simple proof of its conservation. Our considerations here have concerned only the classical theory, but it seems unlikely that a quantal treatment would change anything essential in our conclusions. Adding interaction terms like ϕ^4 or using different matter fields clearly would not change anything either.

In a philosophical debate, examples can be helpful, but in writing this note, I have not attempted to draw any definite conclusions from the example presented above. In particular I have not claimed that the "Kretschmannian" theory formulated herein is necessarily "physically equivalent" to the corresponding special relativistic theory, or that its existence necessarily proves that "general covariance is empty". My primary purpose has been rather to help sharpen such questions by further delineating a simple example some of whose relevant features seem not to be widely known. **

Having done so, however, I would like, in closing, to offer an opinion which I believe to be consistent with the facts presented herein. I believe that general covariance (or perhaps better, *background independence*, since it seems to be true that one can trivially express any theory without exception in generally covariant *language*) is indeed empty if taken by itself, but that it does have meaning if taken in conjunction with a "specification of substance" (and possibly also with other requirements, like the existence of a local Lagrangian). In the case at hand, for example, we had to add a new "substance", namely the field λ^{abcd}, in order to be able to write our action S. Had we stuck with just the fields ϕ and g_{ab}, it is hard to see that we would have been able to promote the latter from the status of background structure to that of dynamical variable, as general covariance demands.

Finally, I would like to thank John Earman for raising some of the questions addressed herein and then encouraging me to write up the answers. I would also like to thank Karel

* As suggested to me by John Earman, a further instructive exercise along these lines might be to carry out a full "Dirac constraint analysis" of (3).

** Many of these same features show up also in certain of the so called "topological quantum field theories". To the extent that people studying such theories typically take for granted that they incorporate general covariance in the physical sense, one may say that, in certain quarters, the debate is effectively over and Kretschmann has won!

Kuchař, John Norton, Don Marolf and Abhay Ashtekar for discussions concerning the implications of some of these results.

Added Note: In working through the example presented herein, I have realized that there may be something special about the case of a Minkowskian background, because the demand for a flat metric is easily expressed in local form through the equation $R^{abcd} = 0$. It is not at all obvious that something similar could be done, even for so simple a geometry as a dust-filled Friedmann universe, let alone for a still less symmetrical background metric. In this sense, it seems an open question how viable the Kretschmann view would have been had we been comparing general relativity with a generically curved background spacetime, rather than with the flat spacetime metric that was historically given to us.

This research was partly supported by NSF grant PHY-0098488, by a grant from the Office of Research and Computing of Syracuse University, and by an EPSRC Senior Fellowship at Queen Mary College. I would also like to express my gratitude to Goodenough College for providing a fine working and living environment during my stay in London where this paper was completed.

APPENDIX: Further details of some of the derivations

In this appendix, I will sketch the derivation of two mathematical facts used above, the first concerning the variation of the "gravity term" in the action-integral (3) and the second being the existence of a potential for any conserved symmetric tensor in flat space.

[*Variation of* $\lambda^{abcd} R_{abcd}$]

For variations of the metric tensor about an originally flat metric (one with vanishing Riemann curvature) one has in general

$$\delta R_{abcd} = \frac{1}{2}(-\partial_{ac}\delta g_{bd} - \partial_{bd}\delta g_{ac} + \partial_{ad}\delta g_{bc} + \partial_{bc}\delta g_{ad}) \ , \tag{8}$$

∂_a being the derivative operator of the flat, unvaried metric and ∂_{ab} being shorthand for $\partial_a \partial_b$. This equation (cf. §92 of [1]) follows straightforwardly from standard formulas expressing δR^a_{bcd} in terms of $\delta \Gamma^a_{bc}$, and $\delta \Gamma^a_{bc}$ in terms of δg_{ab}. Within an overall factor, its right hand side is the only expression with the correct symmetries that one can construct from second derivatives of the metric variation δg_{ab}.

Plugging (8) into the variation of $\lambda^{abcd} R_{abcd}$, integrating by parts twice, and combining with (4) results in equation (5) of the main text.

[*Existence of a potential for* T^{ab}]

Every conserved T^{ab} in Minkowski spacetime can be written as the double divergence of some "potential" H^{abcd} with the symmetries of the Riemann tensor. One can demonstrate this, proceeding along the lines of [2]. Beginning with the conservation equation

$$\partial_b T^{ab} = 0$$

one obtains from the Poincaré lemma a potential W^{abc}, skew in its last two indices, such that

$$T^{ab} = \partial_c W^{abc}$$

The symmetry of T^{ab} then implies that $W^{abc} - W^{bac}$ is divergence free (in the index c) so that, once again, a potential exists:

$$W^{abc} - W^{bac} = \partial_d E^{abcd}$$

where E^{abcd} can be arranged to be skew in both the index pair ab and the index pair cd. Solving this equation for W^{abc} yields

$$T^{ab} = \partial_c W^{abc} = \partial_{mn} \frac{1}{2} (E^{ambn} + E^{bnam})$$

Denoting the argument of ∂_{mn} in this equation by H^{ambn} and replacing it by the combination

$$1/3\ (2H^{ambn} - H^{mban} - H^{bamn})$$

yields an equivalent expression enjoying the full Riemann tensor symmetry. Denoting this new expression by the same letter H, one obtains finally,

$$T^{ab} = \partial_{mn} H^{ambn} \ .$$

References

[1] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, second edition (Addison-Wesley 1962) (translated by Morton Hamermesh)

[2] R.D. Sorkin, "On Stress-Energy Tensors", *Gen. Rel. Grav.* **8:** 437-449 (1977)