Syllabus for Elements of General Relativity I/II (2013)

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1. *Historical place and meaning of General Relativity*
   - Taketani scheme: substance, dynamics and phenomenology
     - (from pattern to substance to dynamics)
   - spacetime as a substance
   - fields as fundamental substances (Faraday)
   - discovery of the metric-field and the causal structure of spacetime
   - fusion of space with time
   - equivalence principle
     - relation of perceived accelerations to the reference frame
     - reduction of gravitation to inertia
   - gauge invariances, symmetries and "general covariance"

2. *Rudimentary Friedmannian cosmology*
   - action-functional for a dust-filled spherical universe
   - effects of a (constant) cosmological constant and of a negative or zero spatial curvature
   - qualitative analysis of the solutions: different universes for different values of the parameters Λ, M and k
   - nature of the initial singularity (embedding diagram)

3. *Rudimentary gravitational kinematics*
   - topology and topological manifolds
   - smooth manifolds, diffeomorphisms and maps of manifolds
   - the metric field expressed in a coordinate chart: its line- and volume- elements
   - tangent vectors and the metric as a tensor field
   - the causal structure of a spacetime

4. *(Non-gravitational) matter in a fixed geometry: how g_{ab} influences other objects*
   - clocks, rulers, light-rays, and reference frames ("observers")
   - particle motion I (analyzed in a coordinate chart)
   - the low-velocity, weak-field approximation
   - wave-motion, the wave equation
   - the $\Delta E \Delta t$ Gedankenexperiment

5. *Some Differential Geometry*
   - tensors and tensor-algebra

abstract indices; vectors, covectors and tensors
bases and components; coordinate bases
the metric tensor again
raising and lowering indices,
orthonormal bases
dimensional analysis for tensors

tensor analysis
the gradient operator
parallel transport and covariant derivative; Christoffel symbols
particle motion in terms of intrinsic quantities
the velocity (tangent) and acceleration vectors of a worldline
geodesic motion as unaccelerated motion

the Lie derivative

Killing vectors (infinitesimal isometries)

derivation of the geodesic equation via intrinsic quantities

6. Particle motion II

conservation laws and Killing vectors

action-functionals for massive and massless particles

thermal equilibrium in a stationary field: the temperature redshift

orbit of a photon in a stationary field: Fermat's principle

particle orbits in the Schwarzschild metric
perihelion shift
light bending
time-delay

7. Dynamics of the gravitational field

Mathematical preparations
the curvature tensor
definition
algebraic symmetry-type
geometric interpretations; geodesic deviation equation
Bianchi identity
contractions of the Riemann tensor: Ricci tensor and Ricci scalar

the gravitational (Hilbert) term in the action-integral S
"derivation" from plausible requirements

how to vary S: the Einstein tensor
how to vary dV, how to integrate by parts

the equations of motion of pure gravity in dimensions 2, 3 and 4

coupling of gravity to other forms of matter: the full Einstein equation
conservation of the right hand side, T^{ab}

8. First applications of the gravitational action-functional and the field equations

- warped product metrics
  - the Riemann and Ricci curvatures
  - curvature of the n-sphere
- the Friedmann universe re-visited (derivation of S_{grav})
- the gravitational field of an isolated spherical object
  - the Schwarzschild solution
  - Birkhoff's theorem

9. Integration and Stokes' theorems

- fluxes and tensor densities
- the relativistic form of a conservation law
- brief introduction to the handout on Stokes' Theorem

10. The stress-energy-momentum tensor T^{ab}

- T^{ab} for the scalar and electromagnetic fields
- the idea of a fluid element
- T^{ab} for a perfect fluid
- general properties of T^{ab}
  - conservation, symmetry, positivity
  - the center of mass and the angular momentum tensor

11. Relativistic perfect fluids

- the fluid equations of motion from energy-momentum conservation
- an action-functional for the Eulerian current vector
  - constraints on the variations of S_{fluid}
- the relativistic superfluid
- causality conditions and the speed of sound

12. Structure of non-rotating stars

- the equations of structure
  - the Einstein tensor of a spherically symmetric metric
  - the equation of hydrostatic equilibrium
  - the "TOV" equation
- exact solution for a incompressible fluid star
- finding solutions numerically in the general case
- mass limits on relativistic stars

13. Cosmology with non-zero pressure and arbitrary curvature-parameter k

- the meaning of the scale factor a when $k = 0$

the equation of state of, and action-functional for the radiation fluid

the "Hamiltonian constraint" or "energy equation"

propagation of light in a Friedmann universe
- particle horizons
- luminosity distance
- observational deduction of the expansion rate and its rate of change

the cosmological parameters, k, Λ, H, ρ, Ω and q
- their inter-relationships and deduction from observations

14. Linearized gravity and gravitational radiation

the linearized metric h_{ab}

the linearized Ricci tensor

gauge conditions

gravitational radiation in empty space

gravitational plane waves
- polarizations

emission of gravitational waves
- the quadrupole formula

Possible further topics

"Global Causal Analysis" and the Positivity of the Total Energy

Black Hole Thermodynamics and Statistical Mechanics

Numerical Relativity

Overview of quantum gravity