Homework set 4, PHY 790, due Thursday, April 4, 2013

14. (a) We defined $\nabla_a f = \partial_a f$ for a scalar field f; and more generally, we defined the covariant derivative of an arbitrary tensor field T at $P \in M$ to be that tensor whose components in a momentarily inertial coordinate chart at P are the partial derivatives of the components of T in the same chart. For example, if ω_a is a covector field then $\nabla_a \omega_b \simeq \partial_a[\underset{(\beta)}{e}{}^c \omega_c]\overset{(\beta)}{e}{}_b$, where the symbol $\simeq$ indicates that equality holds only at the point P at which the basis e is supposed to be inertial.
Prove that ∇_a is "torsion-free": $\nabla_a \nabla_b f = \nabla_b \nabla_a f$ for any scalar field f.
(b) One can define the *Christoffel symbol* with respect to an arbitrary field of basis-vectors by

$$\Gamma^{(\alpha)}_{(\beta)(\gamma)} = \underset{(\gamma)}{e}{}^c (\nabla_c \underset{(\beta)}{e}{}^a)\overset{(\alpha)}{e}{}_a = -\underset{(\gamma)}{e}{}^c (\nabla_c \overset{(\alpha)}{e}{}_a)\underset{(\beta)}{e}{}^a$$

Prove that the second and third expressions are equal. (When we encountered this definition of $\Gamma^{(\alpha)}_{(\beta)(\gamma)}$ in class, $\underset{(\alpha)}{e}{}^a$ was a coordinate-basis, but here we will not assume this. A basis that does not come from a coordinate-chart is sometimes called "non-holonomic".)
(c) Using the definition in (b) show that the components of $\nabla_a \omega_b$ in the basis e are

$$\nabla_{(\alpha)}\omega_{(\beta)} = \frac{\partial \omega_{(\beta)}}{\partial x^{(\alpha)}} - \Gamma^{(\sigma)}_{(\beta)(\alpha)}\omega_{(\sigma)}$$

(d) Using parts (a) and either (b) or (c), prove that the Christoffel symbol is symmetric in its lower indices when it is derived from a coordinate basis.

15. Define the Christoffel symbol as in the previous problem.
(a) Let T_{ab} be a covariant tensor of rank or "valence" 2. Demonstrate that the components of $\nabla_c T_{ab}$ in the basis $\underset{(\alpha)}{e}{}^a$ are given by

$$\partial_{(\gamma)}T_{(\alpha)(\beta)} - \Gamma^{(\sigma)}_{(\alpha)(\gamma)}T_{(\sigma)(\beta)} - \Gamma^{(\sigma)}_{(\beta)(\gamma)}T_{(\alpha)(\sigma)}$$

(b) Apply this to $T_{ab} = g_{ab}$ and use the result together with the metric compatibility equation, $\nabla_c g_{ab} = 0$, to show that when $\underset{(\alpha)}{e}{}^a$ is a *coordinate basis*,

$$\Gamma^{(\alpha)}_{(\beta)(\gamma)} = \frac{1}{2} g^{(\alpha)(\alpha)}\left(g_{(\alpha)(\beta),(\gamma)} + g_{(\alpha)(\gamma),(\beta)} - g_{(\beta)(\gamma),(\alpha)}\right) ,$$

where comma denotes partial derivative. (Recall that in a coordinate basis, the Christoffel symbol is symmetric in its lower indices.) This proves explicitly that the Christoffel symbol defined above is the same as the one we derived earlier (working in a coordinate chart) from the extremality condition $\delta \int_\gamma d\tau = 0$. We see again that "longest curve = straightest curve".

16. A tensor $T^a{}_{bcd}$ satisfies the symmetry conditions

$$T^a{}_{bcd} = -T^a{}_{bdc} \qquad T^a{}_{bcd} + T^a{}_{cdb} + T^a{}_{dbc} = 0$$

and also the trace condition $T^a{}_{abc} = 0$. How many independent components does it have? (In other words, what is dimension of the space of all such tensors?) Find the answer in at least dimensions 1, 2, 3 and 4.

17. (a) Verify by computing Doppler shifts that the redshift between the top and bottom of an accelerating "elevator" in $\mathbb{M}^4$ agrees with what we found earlier in lecture for a weak gravitational field.
(b) Same question for a slowly rotating platform. (In other words compute the redshift between the center and circumference of a rotating disk in two ways: using the Doppler shift in flat spacetime, and using the gravitational redshift formula for the stationary gravitational field that appears when one goes into the frame of reference of the disk.)