Homework set 3, PHY 790, due Thursday, March 14, 2013

10. Let $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ $(\mu, \nu = 0, 1, 2, 3)$ in some coordinate chart, let $\gamma_{ik} = g_{ik} - g_{0i} g_{0k}/g_{00}$ $(i, k = 1, 2, 3)$ be the components of the *spatial metric* in the corresponding reference frame, and let $\gamma^{ik} = g^{ik}$ be the spatial components of the inverse *spacetime* metric.
(a) Show that γ^{ik} and γ_{ik} are inverses of each other.
(b) Find an equation relating $\det(\gamma_{ik})$, $\det(g_{\mu\nu})$, and g_{00}.

11. Consider a "rotating reference frame" in $\mathbb{M}^4$ defined by the family of "observer" world lines

$$\rho = constant \ , \ z = constant \ , \ \phi = \phi_0 + \omega t$$

where t, x, y, z are Cartesian coordinates and $\rho^2 = x^2 + y^2$.
(a) Introduce a convenient set of coordinates for this frame and find the spatial metric $\gamma_{ik} = g_{ik} - g_{0i} g_{0k}/g_{00}$ $(i, k = 1, 2, 3)$ in these coordinates.
(b) Use the result of (a) to compute the ratio of circumference to radius for (spatial) circles in the equatorial plane, centered at the origin. How does your result compare with what you would expect from the Fitzgerald-Lorentz contraction?
(c) Show that clocks cannot be synchronized around such a circle, even though the metric remains time-independent in the rotating frame.
(d) Compute numerical values for the effects in parts (a) and (b) for the earth's equator.
(e) Our reference frame cannot exist physically outside a certain value of ρ. What is this maximum radius?

12. Some simple applications of the formula $dV = \sqrt{|\det g|}\, d^n x$. Three familiar systems of coordinates for three familar flat spaces are (1) polar coordinates (r, θ) for the Euclidean plane, (2) spherical coordinates (r, θ, φ) for Euclidean 3-space, (3) "light-cone" coordinates (u, v) for $\mathbb{M}^2$, where $u = t - x$, $v = t + x$. For each case write the line-element ds^2 in the given coordinates, deduce from it the matrix $g_{\alpha\beta}$, and verify the above formula for dV.

13. Let a, b and c be three vectors (not zero and not parallel) such that $a + b + c = 0$, and consider the three tensors,

$$T_1 = a \otimes a - c \otimes c + a \otimes b - 2b \otimes c$$
$$T_2 = b \otimes b + c \otimes c + b \otimes a + c \otimes b + c \otimes a$$
$$T_3 = -a \otimes a + b \otimes b + c \otimes c$$

(a) Show that $T_1 = T_2 \neq T_3$ (b) Express T_1 in terms of a and b alone.