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Gravity and Cosmology of the dilaton at strong coupling

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Introduction

We live in a epoch of rapid improvements in observational cosmology and experimental gravity. Advances in instrumentation and data processing, in fact, are allowing a very detailed description of the Cosmic Microwave Background anisotropies and conspire to give increasing precision in the evaluation of cosmological parameters such as the age, the spatial curvature, the rate of expansion and of acceleration of the universe. Beside, the direct detection of gravitational waves is expected in the next decade thanks to experiments like the ground based interferometers [GEO600], [VIRGO], [LIGO] and [EURO] and the interferometer space antenna [LISA]. Finally, high precision tests of the equivalence principle will be provided by the missions MICROSCOPE [P. Touboul *et al.*, 2001] and [STEP]. Such a rich observational situation represents an exciting challenge for fundamental physics, opening up a new, "low energy" path to test the theories candidate for unifying all interactions and/or give a quantum description of gravity.

By now, the only known consistent quantum theory of gravity is superstring theory[2] which indeed has been the framework of my work during PhD. In particular, this thesis is based on the work made in collaboration with Gabriele Veneziano, Maurizio Gasperini and Thibault Damour and is devoted to examine possible implications on gravity and cosmology of the string-inspired "strong coupling scenario" that I shortly introduce below.

The low-energy limit of Superstring theory, at tree-level in the string-loop expansion, presents a variety of massless and massive fields including the graviton and hence seems an appropriate starting point to describe all the known interactions in a unified fashion. Moreover, the theory is *finite*, the inverse string length being a UV cut-off for the effective low-energy field theory, and contains no free-parameters, the couplings being related to the vacuum expectation value of the dilaton field. However, despite the presence of a spin-two "gravitational" field, the tree level theory is far from describing gravity as we experience it. The presence of the *dilaton* field and of other long range scalars ("moduli fields") plagues the tree-level theory with unacceptable violations of the equivalence principle.

(*The subject of equivalence principle violations is reviewed in some detail in* Chapter 1 *for a generic theory of gravity in the presence of a single scalar field. Newtonian and Post-Newtonian limits are introduced in order to quantify*

[2] See [Green, Schwarz and Witten, 1987], [Polchinski, 1998]

the deviations from general relativity. The experiments which provide the most strict bounds on such deviations are described and the bounds themselves stated.)

It is common wisdom [Taylor and Veneziano, 1988] to assume that some mass is generated for the moduli fields, thereby suppressing their long range interactions and restoring the equivalence principle on observable scales, say > 1mm. In the case of the dilaton ϕ, the potential that gives it a mass should also *freeze* its value in the so-called "weak coupling region" ($\phi < 0$) and prevent unobserved time-variations of the coupling "constants". However, at weak coupling we should sufficiently trust the perturbative tree-level action where such a potential leaves no trace of itself. Thus, it looks unlikely that non-perturbative effects will be significant enough in the perturbative region to stabilize the moduli. Another possibility is that generic string-loop corrections (at non-weak coupling) are such that the dilaton decouples from the other fields at some value ϕ_m. This possibility has been investigated by [Damour and Polyakov, 1994] who have shown that if such decoupling value exists, it acts as a cosmological attractor for the dilaton and the theory is phenomenologically safe.

Recently, [Veneziano, 2002] has studied the effective gravitational and gauge couplings obtained for a toy quantum-field theory model that resembles the low energy limit of string theory. Thanks to the large number of degrees of freedom that are integrated over, the renormalized couplings tend to be for the most part *induced* by loop corrections and reach an extremum at infinite *bare* coupling. This toy model suggests that the effective action of string theory has in fact a decoupling value ϕ_m for the dilaton but, contrary to the case considered by Damour and Polyakov, this value is at infinity, $\phi_m \to \infty$! Thus, the standard scenario of a dilaton *stuck* at a certain value may change in a new one where the dilaton is running to infinity and still the theory is phenomenologically acceptable. Of course, the point at infinity is much less an efficient cosmological attractor for the dilaton and more consistent equivalence principle (EP) violations are expected.

(*The strategies to get a phenomenologically acceptable theory of gravity from string theory are sketched in* Chapter 2. *Some scales of phenomenological relevance in string theory are also discussed. In the last section (2.3) the basic ideas and motivations of the "strong coupling scenario" are introduced.*)

A primordial inflationary period proves particularly efficient in pushing the dilaton toward its fixed value at infinity, and we have shown [Damour, Piazza and Veneziano, 2002b] that the present value of the dilaton [see equation (3.28) in Chapter 3] can be related to the observed density fluctuations on large scales and to some slow-roll inflationary parameter. Some inflaton's potentials are indeed ruled out in our model by the present experimental bounds on EP violations, since they provide too low a value of the dilaton today. We expect *composition dependent* violations of order $(\Delta a/a) \sim 10^{-12}$, where the parameter $(\Delta a/a)_{AB}$ measures the difference in the acceleration a of two bodies A and B of different composition [Damour, Piazza and Veneziano, 2002a]. Note that the satellite experiments MICROSCOPE [P. Touboul *et al.*, 2001] and [STEP] will check the universality of free-fall up to the levels $(\Delta a/a) \sim 10^{-15}$ and $(\Delta a/a) \sim 10^{-18}$ respectively.

(*In* Chapter 3 *the phenomenology of the model is studied, expecially for what concernes EP violations. The basic references are our works* [Damour, Piazza and Veneziano, 2002a] *and* [Damour, Piazza and Veneziano, 2002b].)

As the dilaton runs to infinity, it may play some relevant role in the evolution of the universe, what a freezed dilaton couldn't do. In particular, we have shown [Gasperini, Piazza and Veneziano, 2002] that, in the presence of some non-perturbative potential, the dilaton may act as a "quintessence" the usually *ad hoc* introduced field that should drive the present accelerated expansion of the universe. Moreover, a non trivial coupling between dilaton and dark matter, which is rather natural in a string context, can lead to a final configuration with fixed positive acceleration and dilatonic over dark matter energy densities ratio. This would consist in a solution of the so-called "cosmic coincidence" problem.

(*In* Chapter 4 *this possible cosmological implications of the strong coupling scenario are discussed in detail. The basic reference is* [Gasperini, Piazza and Veneziano, 2002].)

Contents

Chapter 1

Equivalence Principle violations

A crucial ingredient of General Relativity (GR) is the Equivalence Principle (EP), saying more or less that all bodies fall in a gravitational field with the same acceleration. What in Newtonian gravity may appear just as a "by chance" occurrence (inertial and gravitational masses "happen" to be proportional), by [Einstein, 1907] was called "hypothesis of complete physical equivalence" and put at the basis of the formulation of GR. The aspect emphasized by Einstein is that, in virtue of the universality of free fall, it is always possible to nullify (locally) the effects of the gravitational field by moving to a freely-falling system: a freely-falling observer does not experience gravity just because everything in its neighbourhood falls in the same way as it does. Such a slight change of viewpoint inspired the idea of describing gravity by means of a symmetric "metric" tensor $g_{\mu\nu}$ which can always (locally) take the Minkowski "flat" form by a suitable change of coordinates. By the same token, the effects of gravity are faithfully reproducible by means of accelerated (local) frames.

A posteriori, people have identified three conceptually different "kinds" of equivalence principle. While referring to [Will, 2001] and [Damour, 1996*a*] for a wider and more rigorous treatement of the argument, let's state the three different subprinciples as follows:

- *Weak Equivalence Principle (WEP)*: The trajectory of a test body falling in a gravitational field is independent of its internal composition.

- *Einstenian Equivalence Principle (EEP)*: WEP is true and if you are a freely falling observer the outcomes of your *local non-gravitational experiments* are independent of your velocity and position.

- *Strong Equivalence Principle (SEP)*: WEP is true also for self-gravitating bodies and, if you are a freely falling observer, neither *local gravitational* experiments' outcomes depend on your velocity and position.

Such statements, although conceptually distinct, when embodied in a consistent theoretical framework may result someway related (apart from the trivial SEP $\Rightarrow$ EEP $\Rightarrow$ WEP). In particular, a number of "plausibility" arguments have been given in favour of the so-called *Shiff's conjecture* which asserts that any complete, self consistent theory of gravity that embodies WEP necessarily embodies EEP (WEP $\Rightarrow$ EEP) [Will, 1993]. Thus, at least from a theoretical point of view, it is customary to distinguish only between SEP violations (*composition independent* violations) and WEP-EEP violations (*composition dependent* violations). On the other hand, it is widely believed that a theory of gravity incorporating up to SEP must be a *pure tensor* theory as GR, where gravitation is mediated by - and couples to matter through - the only metric tensor $g_{\mu\nu}$.

One may note that, so far, no deviations from GR have been observed and ask why we should struggle to classify and quantify possible, and still never experienced, EP violations. Beside its appealing on foundational grounds, the argument is of concrete interest in that the theoretical frameworks of modern unification theories, and notaby string theory, suggests that EP must be violated at some level. In particular, all string theory models definitely predict the existence of at least a scalar field, the *dilaton*, and always give a *scalar-tensor* "version" of gravity. It is intriguing that low-energy gravitational physics experiments may discriminate between competing models of unification better than high-energy experiments can actually do!

In this chapter we review EP violations in scalar tensor theories of gravity. In particular, we restrict the discussion to the case in which a single coupled scalar field is present. The generalization to several scalar fields, has been carried out by [Damour and Esposito-Farèse, 1992] and does not present any relevant conceptual novelty respect to the single scalar field case. EP violations in a much wider class of theories of gravity are considered in [Will, 1993] and are beyond the scope of this work.

1.1 Conformally related frames

Suppose we are given the low-energy limit of a supposedly foundamental theory, e.g. a superstring theory model with some dimensional reduction mechanism, in terms of the action

$$S = \frac{\widetilde{M}_*^2}{2}\int d^4x\sqrt{-\widetilde{g}}\,\left[B_g(\phi)\widetilde{R} - B_f(\phi)\,\widetilde{g}^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi\,-\,\widetilde{B}(\phi)\right] + \widetilde{S}_m[\Psi_i,\,\widetilde{g}_{\mu\nu},\,\phi]. \quad (1.1)$$

The mass scale $\widetilde{M}_*$ gives the typical strength of the gravitational coupling to matter and $\widetilde{R}$ is the Ricci scalar constructed with the metric tensor $\widetilde{g}_{\mu\nu}$. The coefficients B_is are assumed to be strictly positive functions of the dimensionless scalar field ϕ. The matter action $\widetilde{S}_m$ contains the "matter" fields Ψ_i that we associate to the standard model of particle interactions. The metric $\widetilde{g}_{\mu\nu}$ is involved in the matter action $\widetilde{S}_m$ through covariant derivatives acting on matter fields.

In order to understand any better the dynamics of (1.1), it is often usefull to bring it into a more familiar form. To begin with, we may want to have the usual Ricci gravitational kinetic term in the action. In $D = 4$ spacetime dimensions this is always possible (see Appendix A for details), by referring to the conformally rescaled *Einstein metric* $g_{\mu\nu}$:

$$\widetilde{g}_{\mu\nu} \longrightarrow g_{\mu\nu} = \frac{\widetilde{M}_*^2}{M_*^2} B_g(\phi)\, \widetilde{g}_{\mu\nu}, \tag{1.2}$$

where M_* is the reduced "naked" Plank mass. The reason why it is "naked" will be explained in the next section. When written in terms of $g_{\mu\nu}$ the gravitational sector of (1.1) reads

$$S_{\text{grav}} = \frac{M_*^2}{2} \int d^4x \sqrt{-g} \left[R - \left(\frac{3}{2} \frac{B_g'^2}{B_g^2} + \frac{B_f}{B_g} \right) g^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi - \frac{M_*^2}{\widetilde{M}_*^2 B_g^2} \widetilde{B}(\phi) \right],$$

where the Ricci scalar R is constructed with the Einstein metric $g_{\mu\nu}$. The unusual kinetic term for ϕ can be easily reduced to a more canonical form by a simple redefinition of the scalar field ϕ:

$$\phi \longrightarrow \varphi = \int d\phi \left(\frac{3}{4} \frac{B_g'^2}{B_g^2} + \frac{1}{2} \frac{B_f}{B_g} \right)^{1/2}, \tag{1.3}$$

bringing action (1.1) to the final "Einstein–Klein Gordon" form

$$S = \frac{M_*^2}{2} \int d^4x \sqrt{-g}\, [R - 2\, g^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi - B(\varphi)] \,+\, S_m[\Psi_i, g_{\mu\nu}, \varphi]\,. \tag{1.4}$$

where $B(\varphi) \equiv M_*^2 \widetilde{M}_*^{-2} B_g^{-2} \widetilde{B}(\phi)$. Although in other contexts a Klein Gordon field is more often defined with mass dimensions, the massless field φ defined in (1.3) is very often used in the litterature on scalar tensor theories of gravity. Note also that here both the original and Einstein-transformed potentials $\widetilde{B}(\phi)$ and $B(\varphi)$ have the unusual dimensions of $[\text{mass}]^2$.

A description in terms of the Einstein metric $g_{\mu\nu}$ is often referred to as *Einstein frame*. Although each frame is mathematically equivalent to any other, in General Relativity we are used to think of the metric as a "physical real object", that gives the physical measurable distances between systems. In a more extended scalar-tensor context as that of (1.4), which is the physical frame? Is there any? Strictly speaking, the answer to the latter is affirmative only in the very special case in which matter couples to gravity in a purely metric way i.e. the case in which the metric $g_{\mu\nu}$ and the scalar field φ enter the matter action through some conformally related metric $\bar{g}_{\mu\nu} = A(\varphi)^2 g_{\mu\nu}$ to which matter universally couples:

$$S_m[\Psi_i,\ g_{\mu\nu},\ \varphi] \;=\; \overline{S}_m[\Psi_i,\ A^2(\varphi) g_{\mu\nu}] \;\equiv\; \overline{S}_m[\Psi_i,\ \bar{g}_{\mu\nu}] \tag{1.5}$$

The "bar" over S_m emphasizes a different functional dependence on its arguments of the "Jordan-frame" matter action $\overline{S}_m$ from the "Einstein-frame" matter action S_m. The above condition guarantees the validity of EEP in the scalar tensor theory we are considering. In fact, by using a simple result about pseudo-Riemannian spaces, one can take any worldline γ and find a local coordinate system along it with respect to which the metric $\bar{g}_{\mu\nu}$ is Minkowskian and has null derivatives. The evolution of a matter system of negligible self-gravity in the vicinity of γ can be described as just that of an isolated system in special relativity and will exhibit no preferred spatial direction nor velocity. Moreover, the local evolution of the system will depend only on the values of the coupling constants and mass scales that enter the usual Standard Model.

The Jordan metric $\bar{g}_{\mu\nu}$, when it exists, is then the *physical* metric, the one measured by rods, clocks, laser interferometers etc ... constructed with the matter fields Ψ_i. Take, say, the platinum-iridium prototype bar kept in Paris: modulo of course some thermodinamical conditions, the bar is always and everywhere 1 "Jordan meter" long, since in the Jordan frame the equations governing its internal structure do not contain the scalar field φ. On the contrary, its "Einstein length" may depend on the local value of φ. So, whenever you measure lengths by means of platinum bars you are actually measuring Jordan frame lengths!

In terms of $\bar{g}_{\mu\nu}$ the action reads

$$S = \frac{M_*^2}{2}\int\sqrt{-\bar{g}}\left[\bar{\phi}\bar{R} - \frac{\omega(\bar{\phi})}{\bar{\phi}}\,\bar{g}^{\,\mu\nu}\,\partial_\mu\bar{\phi}\,\partial_\nu\bar{\phi} - \bar{\phi}^2 B\right] + \overline{S}_m[\Psi_i, \bar{g}_{\mu\nu}]\,, \tag{1.6}$$

where

$$g_{\mu\nu} \longrightarrow \bar{g}_{\mu\nu} = A^2(\varphi) g_{\mu\nu}\,, \tag{1.7}$$

and we have defined

$$\bar{\phi} = A^{-2}(\varphi), \qquad 3 + 2\omega(\bar{\phi}) = A^2(\varphi)/A'^2(\varphi). \tag{1.8}$$

in order to easily recover the original [Brans and Dicke, 1961] theory in the case $\omega = \text{const}$.

So far, we have introduced up to three different conformally related metric tensors: $\widetilde{g}_{\mu\nu}$, $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$. The tilde metric is intended as somehow supplied *ab initio* by our supposedly foundamental quantum theory, whose gravitational phenomenology we want to study. In the case of string theory, $\widetilde{g}_{\mu\nu}$ may well be the metric appearing in the σ-model formulation, the spacetime where perturbative strings live. In this case $\widetilde{M}_*$ in front of action (1.1) can be identified with the string scale M_s which is also the natural cut-off of the low-energy theory (1.1). It is worth saying that, from a stringy point of view, it's right when the "tilde" curvatures ($\widetilde{R}$, $\widetilde{R}^{\mu\nu}\widetilde{R}_{\mu\nu}$) reach the scale M_s that the low energy description breaks down and higher derivative corrections are needed. In any other frame the critical scales for the curvatures generally depend on the local value of the scalar field (the dilaton): from this point of view the "string frame" $\widetilde{g}_{\mu\nu}$ is favored in the high curvature regime. On the other hand, the low curvature, low energy regime is most naturally described in terms of the frame, if it exists, that universally couples to matter, the Jordan frame $\bar{g}_{\mu\nu}$ since it is the one measured by ordinary rods and clocks. It would take a purely gravitational clock, e.g. that defined by the orbital motion of two black holes, to measure the Einstein metric $g_{\mu\nu}$.

In order to study and quantify the deviations from General Relativity of scalar tensor theories we proceed first by considering purely metric couplings and by discussing in this context composition-independent (S)EP violations. Then we go to a wider class of theories to study composition-dependent (W)EP violations.

1.2 Metric couplings and composition-independent EP violations

We now want to study the Newtonian and post-Newtonian limits of the scalar tensor theory (1.1) in the case of a purely metric coupling i.e. when $S_m = S_m[\Psi_i, A^2(\varphi)g_{\mu\nu}]$. Equations are most easily written in the Einstein frame. By varying action (1.4) with respect to $g^{\mu\nu}$ and φ we find

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi - \frac{B(\varphi)}{2}g_{\mu\nu} + \frac{8\pi G_*}{c^4}\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) \tag{1.9}$$

$$\Box\varphi = \frac{B'(\varphi)}{4} - \frac{4\pi G_*}{c^4}\alpha(\varphi)T, \tag{1.10}$$

where the velocity of light c has been explicitely re-inserted for reasons that will be clear in the following, $T_{\mu\nu}$ is the Einstein frame stress-energy tensor,

$$T_{\mu\nu} \equiv -\frac{2c}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}, \qquad T \equiv g_{\mu\nu}T^{\mu\nu}. \tag{1.11}$$

and the box symbol $\Box \equiv \nabla^\mu\partial_\mu$ denotes the usual Einstein-frame Laplacian. Instead of the (bare reduced) Plank mass M_* we have introduced the (bare) Newton constant G_*, where $M_* \equiv (8\pi G_*)^{-1/2}$, and the reason why we call it "bare" is that G_* does not include scalar contributions to the gravitational interaction. The way such contributions enter the game will be clear in the following. Their strength is measured by the crucial parameter

$$\alpha(\varphi) \equiv \frac{d\ln A(\varphi)}{d\varphi}. \tag{1.12}$$

To obtain (1.10) we have made use of assumption (1.5):

$$\frac{\delta S_m}{\delta\varphi(x)} = \int d^4y\, \frac{\delta\overline{S}_m[\Psi_i, \bar{g}_{\mu\nu}]}{\delta\bar{g}^{\mu\nu}(y)}\,\frac{\delta\bar{g}^{\mu\nu}(y)}{\delta\varphi(x)}, \tag{1.13}$$

where the first factor in the integral defines the Jordan-frame stress-energy tensor in a way similar to (1.11):

$$\overline{T}_{\mu\nu} \equiv -\frac{2c}{\sqrt{-\bar{g}}}\frac{\delta S_m}{\delta\bar{g}^{\mu\nu}} = A^{-2}(\varphi)T_{\mu\nu}, \qquad \overline{T} \equiv \bar{g}_{\mu\nu}\overline{T}^{\mu\nu} = A^{-4}(\varphi)T. \tag{1.14}$$

1.2.1 Newtonian limit and effective Newton constant

Our first task is to determine the effective gravitational force acting between two bodies according to the scalar-tensor theory (1.1). For this purpose, we analyze the *Newtonian limit* of the theory which consists of two dinstinct steps: (*i*) linearizing Einstein's equations around a Minkonski space time and (*ii*) neglecting velocity-dependent terms in the linearized equations. In other words we are assuming that, as in the solar system, typical relative velocities are small compared to the velocity of light. This allows to refer to a common Minkowskian background and to neglect velocity dependent terms referred to such background. We also assume that curvatures are everywhere small on Plank scales so that we can trust linearized equations.

So we linearize equations (1.9) and (1.10) around a flat Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and a constant scalar field configuration φ_0:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \varphi \longrightarrow \varphi_0 + \varphi. \tag{1.15}$$

The first order Christoffel symbols and Ricci tensor components read

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}) \tag{1.16}$$

$$\begin{aligned} R_{\mu\nu} &= \partial_\rho \Gamma^{\rho}_{\mu\nu} - \partial_\mu \Gamma^{\rho}_{\rho\nu} \\ &= \frac{1}{2}(-\Box h_{\mu\nu} + \partial_\rho\partial_\nu h^{\rho}_{\mu} - \partial_\mu\partial_\nu h^{\rho}_{\rho} + \partial_\mu\partial_\rho h^{\rho}_{\nu}). \end{aligned} \tag{1.17}$$

where now $\Box$ is the first order Minkowskian Laplacian $\Box = \partial^\rho\partial_\rho$ and indices are raised and lowered with the Minkowski metric $\eta_{\mu\nu}$, as usual in linear theory [Wald, 1984]. It is known that in the linearized approach the components of $h_{\mu\nu}$ – as well as those of $g_{\mu\nu}$ in the full GR formalism – change under a coordinate transformations. This gauge freedom can be fixed by choosing the *harmonic gauge*, defined by the relation

$$\partial^\nu h_{\mu\nu} = \frac{1}{2}\partial_\mu h \qquad \text{(harmonic gauge)}.$$

In the harmonic gauge equations (1.9) and (1.10) assume the final linear form

$$\Box h_{\mu\nu} = -\frac{16\pi G_*}{c^4}\left(T_{\mu\nu} - \frac{1}{2}T\eta_{\mu\nu}\right), \tag{1.18}$$

$$(\Box - m^2)\varphi = -\frac{4\pi G_*}{c^4}\,\alpha(\varphi_0)T. \tag{1.19}$$

In going from (1.9) and (1.10) to (1.18) and (1.19) we have also made the reasonable assumption that the background value of the dilaton, φ_0 resides at a minimum of its potential, $B'(\varphi_0) = 0$. Moreover, to avoid a cosmological constant term in the equations, we have also assumed that $B(\varphi_0) = 0$. Finally we have defined the constant m through the second derivative of B in φ_0 :

$m^2 \equiv B''(\varphi_0)/4$. In what follows we denote for brevity with α the value of $\alpha(\varphi)$ at the minimum of the potential: $\alpha \equiv \alpha(\varphi_0)$.

The above equations have the two (formal) solutions

$$h_{\mu\nu} = -\frac{16\pi G_*}{c^4}\,\Box^{-1}\left(T_{\mu\nu} - \frac{1}{2}T\eta_{\mu\nu}\right), \tag{1.20}$$

$$\varphi = -\frac{4\pi G_*}{c^4}\,\alpha\,(\Box - m^2)^{-1}T. \tag{1.21}$$

We now move to the physical Jordan frame and study the interaction acting between two bodies. To begin with, we need the gravity–matter interaction term at the linearized level. Following [Damour, 1996*a*], we note that the Jordan-frame analogue of (1.11) can be written as

$$\delta S_m = \frac{1}{2c}\int d^4x\sqrt{-\bar{g}}\;\overline{T}^{\mu\nu}\;\delta\bar{g}_{\mu\nu}$$

which is an interaction term of the required order in the perturbations, since the background value of the stress-energy tensor is null. This suggests that gravity couples to the physical stress energy tensor $\overline{T}_{\mu\nu}$ through the perturbed Jordan metric

$$\delta\bar{g}_{\mu\nu} = \delta[A^2(\varphi)g_{\mu\nu}] = A(\varphi_0)^2(h_{\mu\nu} + 2\eta_{\mu\nu}\alpha\,\varphi) + \text{higher order terms},$$

so that the linearized interaction action is

$$S_{\rm int} = \frac{A(\varphi_0)^2}{2c}\int d^4\bar{x}\,(h_{\mu\nu} + 2\,\alpha\,\varphi\,\eta_{\mu\nu})\,\overline{T}^{\mu\nu}. \tag{1.22}$$

Note in fact that at zeroth order in the perturbations, the conformal rescaling (1.7) brings the background Minkowski spacetime into itself. In (1.22) and in the following we deal with this fact by means of a coordinate rescaling $x^\mu \to \bar{x}^\mu = A(\varphi_0)x^\mu$ such that the Jordan – "rescaled" Minkowski metric $\bar{\eta}_{\mu\nu}$ reads $\bar{\eta}_{\mu\nu} = \text{diag.}(-1,1,1,1)$ only when written in terms of the new coordinates $\bar{x}^\mu$. Thus, the generally covariant rescaling of the volume element $d^4x\sqrt{-\bar{g}} = d^4x\sqrt{-g}\,A^4(\varphi)$ reduces to

$$d^4x \longrightarrow d^4\bar{x} = A^4(\varphi_0)d^4x,$$

and the *Jordan frame*- linearized Laplacian $\bar{\Box}$ is defined by

$$\bar{\Box} \equiv \bar{\eta}_{\mu\nu}\frac{\partial}{\partial\bar{x}^\mu}\frac{\partial}{\partial\bar{x}^\nu} = A^{-2}(\varphi_0)\Box\;. \tag{1.23}$$

In order to study the gravitational interaction of two localized objects, one "here" and the other "there" with physical stress-energy tensors ${}_{\rm h}\overline{T}^{\mu\nu}$ and ${}_{\rm t}\overline{T}^{\mu\nu}$ respectively, we write with the aid of (1.22) the interaction between the body "here" and the gravitational fields, say ${}_{\rm t}h_{\mu\nu}$ and $\varphi_{\rm t}$, produced by "there":

$$S_{\rm int} = \frac{A(\varphi_0)^2}{2c}\int d^4\bar{x}\,(\,{}_{\rm t}h_{\mu\nu} + 2\,\alpha\,\varphi_{\rm t}\eta_{\mu\nu})\;{}_{\rm h}\overline{T}^{\mu\nu}. \tag{1.24}$$

We can then substitute ${}_{\mathrm{t}}h_{\mu\nu}$ and φ_{t} with their solutions given by (1.20) and (1.21) and note that the various products do not change in form by going to the Jordan frame, e.g. $\Box^{-1}T_{\mu\nu} = \bar{\Box}^{-1}\overline{T}_{\mu\nu}$. We obtain the interaction action directly in terms of the stress energy tensors of the two bodies:

$$S_{\mathrm{int}} = -A(\varphi_0)^2 \frac{4\pi G_*}{c^5} \int d^4\bar{x} \left[{}_{\mathrm{h}}\overline{T}^{\mu\nu}\, \bar{\Box}^{-1} \left(2\, {}_{\mathrm{t}}\overline{T}_{\mu\nu} - {}_{\mathrm{t}}\overline{T}\bar{\eta}_{\mu\nu}\right) + \alpha^2\, {}_{\mathrm{h}}\overline{T}(\bar{\Box} - \bar{m}^2)^{-1}\, {}_{\mathrm{t}}\overline{T} \right]. \quad (1.25)$$

Equation (1.25) shows clearly that the interaction between "here" and "there" is mediated by a spin-2 field and a possibly massive spin-0 field.

To obtain the *Newtonian limit* we go one step further and assume that the velocities of the two bodies are small. In the limit of two quasi- static point-like objects of masses M_1 and M_2

$$\overline{T}^{\mu\nu} \sim \bar{\eta}^{\mu 0}\bar{\eta}^{\nu 0} M \delta^3(\mathbf{x} - \mathbf{x_0}), \quad T \sim -M\delta^3(\mathbf{x} - \mathbf{x_0}) \quad (1.26)$$

the Langrangian (not "Lagrangian density"!) is

$$L_{\mathrm{int}}(\mathbf{r}) = -4\pi G_* A(\varphi_0)^2 M_1 M_2 \int d^3x\, \delta^3(\mathbf{x} - \mathbf{r}) \left[\triangle_x^{-1} + \alpha^2(\triangle - m^2)_x^{-1}\right] \delta^3(\mathbf{x}) \quad (1.27)$$

where we have put one of the two bodies at the origin of the coordinate frame $\mathbf{x} = 0$ and the other at the point $\mathbf{r}$. We have also neglected the time-derivatives in the D'Alambertian, introduced the flat-space Laplacian $\triangle$ and omitted the "bar" over the physical quantities for readability. By Fourier transforming into momentum space we obtain

$$\begin{aligned} L_{\mathrm{int}}(\mathbf{r}) &= -4\pi G_* A(\varphi_0)^2 M_1 M_2 \int d^3x \int \frac{d^3p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{r})} \,\cdot \\ &\quad \cdot \int \frac{d^3k}{(2\pi)^3} \left(\frac{1}{-\mathbf{k}^2} + \frac{\alpha^2}{-\mathbf{k}^2 - m^2} \right) e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= 4\pi G_* A(\varphi_0)^2 M_1 M_2 \int \frac{d^3k}{(2\pi)^3} \left(\frac{1}{-\mathbf{k}^2} + \frac{\alpha^2}{-\mathbf{k}^2 - m^2} \right) e^{i\mathbf{k}\cdot\mathbf{r}} \,. \end{aligned}$$

It is sufficient to calculate the integral for the second term in the brackets since the first one is a particular case ($\alpha = 1$, $m = 0$) of the second. This ammounts to deriving the Yukawa potential by Fourier transforming the propagator of a massive particle. By using spherical coordinates in the momentum space $k \equiv |\mathbf{k}|, \theta$ we get

$$\begin{aligned} \int \frac{d^3k}{(2\pi)^3} \frac{\alpha^2}{\mathbf{k}^2 + m^2} e^{i\mathbf{k}\cdot\mathbf{r}} &= \frac{\alpha^2}{(2\pi)^2} \int_0^\infty dk \frac{k^2}{k^2 + m^2} \int_0^\pi d\theta \sin\theta\, e^{ikr\cos\theta} \\ &= \frac{\alpha^2}{ir(2\pi)^2} \int_{-\infty}^{\infty} dk k \frac{e^{ikr}}{k^2 + m^2} = \frac{\alpha^2}{4\pi} \frac{e^{-mr}}{r}. \end{aligned}$$

In the last step we have close the contour of the integral above in the complex plane picking up the residue of the simple pole at $k = +im$. Putting all together we eventually find

$$L_{\rm int}(r) \;=\; -V_{\rm grav}(r) \;=\; A(\varphi_0)^2 G_* M_1 M_2 \left(\frac{1}{r} + \frac{\alpha^2 e^{-mr}}{r}\right). \tag{1.28}$$

As expected, in the presence of a massive scalar field, the usual $1/r$ gravitational potential gets modified by a Yukawa-type term that dies out at a distance $\lambda \simeq 1/m$.

Modifications of the $1/r$ scaling of gravity have been experimentally constrained over distances above the centimeter. The bounds on the Yukawa-type term in (1.28) are [Damour, 1996*a*]

$$\begin{array}{lll} |\alpha|^2 \lesssim 10^{-4} & \text{if} & \lambda \simeq 1 \text{ cm} \\ |\alpha|^2 \lesssim 5\times 10^{-4} & \text{if} & \lambda \simeq 1 \text{ m} \\ |\alpha|^2 \lesssim 10^{-3} & \text{if} & 10 \text{ m} \lesssim \lambda \lesssim 10 \text{ km} \\ |\alpha|^2 \lesssim 10^{-8} & \text{if} & 10^4 \text{ km} \lesssim \lambda \lesssim 10^5 \text{ km} \end{array}$$

Quite surprisingly, sub-millimeter scales are very poorly constrained and are currently the subject of experimental tests [Long, Churnside and Price, 2000]. These short-distance gravity experiments are highly desirable, also because modifications of gravity under the millimeter (with a different power-law term of the kind $1/r^n$) have been recently predicted by new models of compactifications such as that of [Arkani-Hamed, Dimopoulos and Dvali, 1998].

The two main routes to a phenomenologically accettable theory of gravity from string-inspired models are already suggested by the eloquent expression (1.28): either you suppose that the dilaton have a sufficiently large mass (say, $m > m_0$) to suppress its long-range interactions [Taylor and Veneziano, 1988], or you assume to find yourself in a region of the parameter space of the theory where the dilaton is sufficiently decoupled (say $\alpha < \alpha_0$) [Damour and Polyacov, 1994], [Damour, Piazza and Veneziano, 2002a].

In what follows we shall concentrate on the case of a long-ranged scalar field, $m = 0$. In this case (1.28) predicts a $1/r$ potential between two masses and an *effective Newton constant* given by

$$G(\varphi) \;=\; G_*\, A(\varphi)^2\, [1 + \alpha(\varphi)^2]. \tag{1.29}$$

The Einstein frame Newton constant G_* is rescaled by a factor $A(\varphi)^2$, as already evident in the Jordan-frame action (1.7) where we recall that $\bar{\phi}$ is right $A(\varphi)^{-2}$, and gets a corrective contribution by the scalar field interactions proportional to the parameter $\alpha(\varphi)^2$ defined in (1.12). This can provide deviations form GR in two related ways.

First, the Newton "constant" may vary in space and time depending on the background (or vacuum expectation) value of φ, so that by performing gravitational experiments in different space-time regions one may find different outcomes. This constitutes, by the very definition at the beginning of this chapter, a violation of SEP. In a homogeneous and isotropic universe one may expect the "background" dilaton to be roughly constant on spatial slices of constant cosmological time t = const. and significant variations on *Hubble time scales*, $H_0^{-1} \simeq 10^{10}$ yr. Unfortunately, the solar system observations and experiments providing the best constraints on the time variation of G do not significantly restrict such a rough estimate, giving limits of order $|\dot{G}/G| \lesssim 10^{-11}$ yr^{-1}. A complete account of the experimental bounds on $|\dot{G}/G|$ can be found in [Uzan, 2002].

Second, if the "background" value of φ was fixed by some mechanism, one could never experience deviations from GR in the stationary Newtonian limit, since, by (1.29) the effect of a massless scalar field would correspond to an unobservable rescaling of the Newton constant. Still, there are *post-Newtonian* effects that discriminate GR from pure-metrically coupled theories of gravity. Such post-Newtonian effects put the more severe phenomenological constraints on the existence of long-range scalar fields and are the following subject of study.

1.2.2 Post-Newtonian limit and deviations form General Relativity

One may note that in the Newtonian limit, apart from the dilaton, only the deviations from Minkowski of the time-time components g_{00} and T^{00} are considered. As long as bodies are quasi-stationary, in fact, only T^{00} is different from zero and, by equation (1.20) and (1.21) with $m = 0$,

$$g_{00} = -1 + \frac{2\,U_{\rm New}(x)}{c^2}, \quad g_{ij} = \delta_{ij}, \quad g_{0i} = 0, \tag{1.30}$$

$$\varphi_{\rm New} \;=\; -\,\alpha\,\frac{U_{\rm New}(x)}{c^2} \tag{1.31}$$

where $U_{\rm New}(x)$ is the Newton potential for a generic quasi-static matter distribution:

$$U_{\rm New}(\mathbf{x},t) \;=\; \frac{G_*}{c^2}\int d^3x'\;\frac{T^{00}(\mathbf{x}',t)}{|\mathbf{x}'-\mathbf{x}|}\,, \tag{1.32}$$

and, consistently with the quasi-static assumption of Newtonian approximation, we have neglected the time derivatives in the D'Alambertian.

The Newtonian limit can then be viewed as an expansion around Minkowski where the metric components and the dilaton fluctuations φ from the background value φ_0 are considered up to the following orders in $1/c$:

$$g_{00},\ \varphi \;\rightarrow\; \mathcal{O}(c^{-2}), \qquad\qquad g_{i0},\ g_{ij} \;\rightarrow\; \mathcal{O}(c^{0}),$$

while the only non-null stress-energy tensor component T^{00} is a quantity of order c^2 :

$$T^{00} = \mathcal{O}(c^2).$$

While referring always to slowly varying, weakly interacting gravitational system, we can make a step further and consider the next order in $1/c$ of the above quantities i.e. the *P*ost-Newtonian order. Note that this imply to consider in the equations both the linear velocity dependent terms that we neglected in the Newtonian approximation, and some terms beyond the linear order. At first post-Newtonian (1PN) level we need to treat the gravitational quantities up to the following order:

$$g_{00},\ \varphi\ \rightarrow\ \mathcal{O}(c^{-4}), \qquad g_{i0}\ \rightarrow\ \mathcal{O}(c^{-3}), \qquad g_{ij}\ \rightarrow\ \mathcal{O}(c^{-2})$$

while the stress energy tensor components are of the following order

$$T^{00} = \mathcal{O}(c^2), \qquad T^{i0} = \mathcal{O}(c), \qquad T^{ij} = \mathcal{O}(c^0)\,.$$

In what follows we use a Post-Newtonian formalism developed by [Blanchet, Damour and Schafer,1990] and applied to the several scalar field case by [Damour and Esposito-Farèse, 1992]. The main advantages of this formalism are the use of a convenient parametrization of the metric components that greatly simplify the equations and the possibility to consider any type of matter content and not only a perfect fluid. Still, differently from [Damour and Esposito-Farèse, 1992], we won't use the harmonic gauge but the more standard *Post-Newtonian gauge* [Will, 1993] defined by the relations

$$\begin{cases} h_{0k,k} = \frac{1}{2}h_{,0} \\ \\ h_{ik,k} = \frac{1}{2}(h_i - h_{00,i}) \end{cases} \qquad (h \equiv h_{kk}) \tag{1.33}$$

To write the equations (1.9) and (1.10) at 1PN level we need to express $R_{\mu\nu}$ to the required order in $h_{\mu\nu}$. We have:

$$\begin{cases} R_{00} = -\frac{1}{2}\nabla^2 h_{00} - \frac{1}{2}|\nabla h_{00}|^2 + \frac{1}{2}h_{kl}h_{00,kl} \ + \ \mathcal{O}(c^{-6}) \\ \\ R_{0i} = -\frac{1}{2}(\nabla^2 h_{0i} + \frac{1}{2}h_{00,i0}) \ + \ \mathcal{O}(c^{-5}) \\ \\ R_{ij} = -\frac{1}{2}\nabla^2 h_{ij} \ + \ \mathcal{O}(c^{-4}) \end{cases} \tag{1.34}$$

In the exponential parametrization of [Blanchet, Damour and Schafer,1990] the metric components are expressed in terms of the potentials $U(x)$ and $A_i(x)$ as follows

$$g_{00} \ = \ -\exp\left(-\frac{2U}{c^2}\right) \ = \ -1 + \frac{2}{c^2}U - \frac{2}{c^4}U^2 + \mathcal{O}(c^{-6}) \tag{1.35}$$

$$g_{0i} \ = \ -\frac{A_i}{c^3} + \mathcal{O}(c^{-5}) \tag{1.36}$$

$$g_{ij} \ = \ \delta_{ij}\exp\left(+\frac{2U}{c^2}\right) \ = \ \delta_{ij}\left(1 + \frac{2U}{c^2}\right) + \mathcal{O}(c^{-4}) \tag{1.37}$$

from which follows

$$\sqrt{-g} = \exp\left(-\frac{2U}{c^2}\right) + \mathcal{O}(c^{-6}) \tag{1.38}$$

$$g^{00} = -\exp\left(+\frac{2U}{c^2}\right) + \mathcal{O}(c^{-6}) \tag{1.39}$$

$$g^{0i} = -\frac{A_i}{c^3} + \mathcal{O}(c^{-5}) \tag{1.40}$$

$$g^{ij} = \delta_{ij}\exp\left(-\frac{2U}{c^2}\right) + \mathcal{O}(c^{-4}) \tag{1.41}$$

Note that we have just moved a step forward equation (1.30) and that the potential U needs now to be considered up to order c^{-2}. By defining the symbol "$\approx$" as "equal to the required order in $1/c$" we can calculate the contravariant components of the Ricci tensor in terms of the potentials U and A_i the remarkably simple form of R^{00} being the main advantage of the "exponential parametrization" we are using.

$$\begin{cases} R^{00} \approx g^{00}g^{00}R_{00} \approx e^{4U/c^2}\dfrac{\nabla^2 U}{c^2}\left(\dfrac{4U}{c^2}-1\right) \approx -\dfrac{\nabla^2 U}{c^2} \\ R^{0i} \approx \eta^{00}\eta^{ij}R_{0j} \approx \dfrac{1}{2c^2}U_{,0i} - \dfrac{1}{2c^3}\nabla^2 A_i \\ R^{ij} \approx \eta^{i\alpha}\eta^{j\beta} = -\delta_{ij}\dfrac{\nabla^2 U}{c^2} \end{cases} \tag{1.42}$$

In order to write the components of the tensor on the RHS of (1.9) we note that

$$T = T^{\mu\nu}(\eta_{\mu\nu}+h_{\mu\nu}) = -T^{00} + (T^{00}h_{00}+T^{kk}) + (2T^{0i}h_{0i}+T^{ij}h_{0i}) + \ldots$$

where we have grouped together in parenthesis terms of the same order in $1/c$. We obtain

$$\begin{cases} T^{00} - \dfrac{1}{2}Tg^{00} \approx \dfrac{1}{2}(T^{00}+T^{kk}) \\ T^{0i} - \dfrac{1}{2}Tg^{0i} \approx T^{0i} \\ T^{ij} - \dfrac{1}{2}Tg^{ij} \approx \dfrac{1}{2}T^{00}\delta_{ij} \end{cases} \tag{1.43}$$

The equation for the components (ij) is nothing more than the c^{-2} order equation (1.32). The other equations are:

$$\nabla^2 A_i = -\frac{16\pi G_*}{c}T^{0i} + c\,U_{,0i} \tag{1.44}$$

$$\nabla^2 U = -\frac{4\pi G_*}{c^2}(T^{00} + T^{kk}) \tag{1.45}$$

The quantity $T^{00} + T^{kk}$ plays the role of an "active gravitational mass density", what in more standard Post Newtonian formalisms is generally expressed as a cumbersome combination of perfect-fluid parameters. In order to write the equation for the dilaton at the required level we have to pay some attention in going from the full-covariant D'Alambertian $\Box$ of equation (1.10) to the flat one $\Box_\eta = \eta^{\mu\nu}\partial_\mu\partial_\nu$ appearing already in linearized equations (1.18) and (1.19):

$$\begin{aligned} \Box &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) \approx \frac{1}{\sqrt{-g}}\eta^{\mu\nu}\partial_\mu\partial_\nu \\ &\approx e^{-2U/c^2}\Box_\eta \end{aligned} \tag{1.46}$$

Equation (1.10) at 1PN level thus reads

$$\Box_\eta \varphi = \frac{4\pi G_*}{c^4}\,\alpha(\varphi)(T^{00} - T^{kk}) \tag{1.47}$$

where 1PN corrections to (1.19) are given by the subleading term T^{kk} and by $\alpha(\varphi)$ that must now be considered up to order c^{-2} i.e. calculated at φ_{New} the lowest order solution given in (1.31). The solutions of (1.45) and (1.47) thus read

$$U(\mathbf{x}, t) = \frac{G_*}{c^2}\int d^3x' \,\frac{T^{00}(\mathbf{x}', t) + T^{kk}(\mathbf{x}', t)}{|\mathbf{x}' - \mathbf{x}|}\,, \tag{1.48}$$

$$\varphi(\mathbf{x}, t) = \frac{1}{c^2}\int d^3x' \,\frac{F(\mathbf{x}', t)}{|\mathbf{x}' - \mathbf{x}|} \tag{1.49}$$

where the source term for the dilaton contains also the lower-order solution φ_{New}:

$$F(\mathbf{x}', t) \equiv \frac{1}{4\pi}\frac{\partial^2}{\partial t^2}\varphi_{\mathrm{New}} + G_*\alpha(\varphi_{\mathrm{New}})(T^{00} - T^{kk}). \tag{1.50}$$

We now move to the Jordan physical frame $\widetilde{g}_{\mu\nu} = A^2(\varphi)g_{\mu\nu}$. Consistently with 1PN order we expand the conformal factor as

$$A^2(\varphi) = A^2(\varphi_0)\,\exp(2\alpha\varphi + \bar{\beta}\varphi^2) + \mathcal{O}(c^{-6}) \tag{1.51}$$

where the parameter $\bar{\beta}$ is defined as

$$\bar{\beta} = \frac{d^2A(\varphi)}{d\varphi^2} = \frac{d\,\alpha(\varphi)}{d\varphi}\,. \tag{1.52}$$

By introducing the Jordan-frame potential

$$\widetilde{U} \equiv U - c^2\alpha\varphi + \mathcal{O}(c^{-4}) = (1 + \alpha^2)\,U_{\mathrm{New}} + \mathcal{O}(c^{-2}) \tag{1.53}$$

we get, for the physical metric $\widetilde{g}_{\mu\nu}$ rescaled by an un-physical constant number,

$$\widetilde{g}_{00} \approx -\exp\left(-\frac{2\widetilde{U}}{c^2}+\frac{2(\beta-1)\widetilde{U}^2}{c^4}\right) \approx -1+\frac{2\widetilde{U}}{c^2}-\frac{2\beta\widetilde{U}^2}{c^4}, \tag{1.54}$$

$$\widetilde{g}_{0i} \approx -\frac{2}{c^3}(\gamma+1)A_i, \tag{1.55}$$

$$\widetilde{g}_{ij} \approx \delta_{ij}\exp\left(\frac{2\gamma\widetilde{U}}{c^2}\right) \approx \delta_{ij}\left(1+\frac{2\gamma\widetilde{U}}{c^2}\right). \tag{1.56}$$

In the above equations we have made use of relation (1.31), and we have also introduced the two crucial parameters β and γ that characterize the 1PN limit of a scalar-tensor theory of gravity. They coincide with the parameters introduced by Eddington long ago when considering the gravitational field of one central massive body($U \sim M/r$) in a Brans-Dicke theory. General Relativity corresponds to the case $\beta = \gamma = 1$ (although for $U \sim M/r$ the usual Schwarschild solution is not recovered because of the Post Newtonian gauge we have choosen). In particular, it turns out that $\gamma - 1$ and $\beta - 1$ measure the amount of non-general relativistic velocity-dependent and non-linear terms respectively in going from Newtonian to Post-Newtonian approximation. In deriving the equation above one finds the relation

$$\gamma - 1 = -2\frac{\alpha^2}{1+\alpha^2} \tag{1.57}$$

$$\beta - 1 = \frac{\alpha^2\bar{\beta}}{2\,(1+\alpha^2)^2}. \tag{1.58}$$

Equation (1.48), written in terms of Jordan-frame quantities, gets modified into

$$\Box\,\widetilde{U} = -4\pi G(\varphi)\left[1+(3\gamma-2\beta-1)\frac{\widetilde{U}}{c^2}\right]\frac{\widetilde{T}^{00}+\widetilde{T}^{kk}}{c^2} \tag{1.59}$$

where $\widetilde{T}^{\mu\nu} = A^{-6}(\varphi)T^{\mu\nu}$ is the Jordan frame stress-energy tensor and G is the effective Newton constant given in (1.29).

In the single scalar field case that we are considering, deviations form General Relativity are encoded in the two Post-Newtonian parameters β and γ. Once predictions for the behaviour of matter are obtained in terms of β and γ, one can compare such predictions with experiments.

1.2.3 Measuring γ

The experiments considered in the following are made with light rays travelling in the gravitational field of the Sun. Since the slow velocity approximation is not appropriate in this case we have to treat space and time on the same footing i.e. all metric components must be considered at the same order in $1/c$. Then, up to order c^{-2}, by equations (1.54)–(1.56), experiments with light rays can only constraint the parameter γ.

The first test we consider was proposed by [Shapiro, 1964] and involves the *time delays* between transmission of radar pulses toward either of the inner planets (Venus or Mercury) and detection of the echos. At order c^{-2}, in fact, the metric (1.54)–(1.56) is diagonal, and along a null ray $ds = 0$ we have

$$dt^2 = \frac{\widetilde{g}_{ii}}{c^2\,\widetilde{g}_{00}}\,dl^2. \tag{1.60}$$

By substituting (1.54) and (1.56) and integrating we obtain

$$c\,\Delta t = \int |d\,\mathbf{l}|\left[1 + (\gamma+1)\frac{\widetilde{U}}{c^2}\right]. \tag{1.61}$$

where $\mathbf{l}$ is the vector tangent to the trajectory of the light ray. The total time of travel of a light ray is delated according to the strength of the gravitational potential encountered along its path. This effect is maximum for a light ray which just grazes the Sun. This is the case when the target planet is at "superior conjunction", i.e. on the opposite side of the Sun from the Earth. By substituting the Newtonian potential of a pointlike object of mass M, $\widetilde{U}(\mathbf{r}) = GM/|\mathbf{r}|$ we then estimate, with respect to the flat space case, a time delay of

$$c\,\delta t \simeq \frac{GM}{c^2}(1+\gamma)\left[2\int_0^{l_+}\frac{dx}{\sqrt{d^2+x^2}} + 2\int_0^{l_p}\frac{dx}{\sqrt{d^2+x^2}}\right], \tag{1.62}$$

where $l_+ \simeq 1.5\times 10^{13}$cm and l_p are the Earth-Sun and planet-Sun distances respectively, and d is the distance of closet approach of the light ray to the center of the Sun. In the limit $l_+ \gg d$, $l_p \gg d$, the above integrals give

$$\delta t \simeq 2\frac{GM}{c^3}(1+\gamma)\ln\frac{4\,l_p\,l_+}{d^2}. \tag{1.63}$$

The expected order of magnitude of the time delay effect is given by the Schwarzchild radius of the Sun in time units $2GM/c^3 \simeq 10^{-5}\,\mathrm{s.} = 10\,\mu\mathrm{s}$. It is usefull to introduce the physical radius of the Sun $R_s \simeq 7\times 10^{10}$ cm and express the logarithm in equation (1.63) as

$$\ln\frac{4\,l_p\,l_+}{d^2} = \ln 4 + \ln\left(\frac{l_+}{R_s}\right)^2 + \ln\left(\frac{R_s}{d}\right)^2 + \ln\left(\frac{l_p}{l_+}\right). \tag{1.64}$$

The second term on the RHS of eq. (1.64) is known (~ 10.6) while d and l_p in the last two terms are specific of the experiment and are meaningfully expressed as ratios of the radius of the Sun and of the so-called astronomical unit l_+ respectively. We obtain finally

$$\delta t \simeq \frac{1+\gamma}{2}\left[240\,\mu\mathrm{s} - 20\,\mu\mathrm{s}\ln\left(\frac{d}{R_s}\right)^2\left(\frac{l_+}{l_p}\right)\right]. \tag{1.65}$$

Since Shapiro's discovery of this effect a number of measurements have been made using ranging to targets passing through superior conjunction. Since one does not have access to a "Newtonian – flat space" signal against which to compare the round trip travel time of the observed signal, it is necessary to do a differential measurement of the variations in round trip travel times as the target passes through superior conjunction, and to look for the logarithmic behaviour (1.65). Up to date, the most precise measurement of the coefficient $(1+\gamma)/2)$ has been performed in 1979 by the Viking Mars landers and orbiters, used as active retransmitters of the radar signals. The final result of the Viking time-delay experiment is [Reasenberg et al. , 1979]

$$\frac{1+\gamma}{2} = 1 \pm 0.001\,. \tag{1.66}$$

For more than a decade (1.66), provided by the time-delay effect, has been the most tight constraint on γ. The other celebrated effect of *light deflection* by the Sun, which was one of the first great successes of Einstein General Relativity, could not give great accuracies, at least in its most classical version of optical lightstar observations during a solar eclipse. It can be shown [Will, 1993] that at 1PN order the deflection angle $\delta\theta$ of a light ray passing near the Sun is given by

$$\delta\theta = \frac{1+\gamma}{2}\frac{2GM}{c^2 d}\,(1+\cos\theta_0), \tag{1.67}$$

where θ_0 is the "Newtonian–flat spacetime" angle between the source and the center of the Sun. The derivation of (1.67), although straightforward, is a bit more complicate than that of (1.65) and won't be presented here. As expected, the maximum effect is obtained when the light ray passes very near the Sun: $d \simeq R_s$, $\cos\theta_0 \simeq 1$ and

$$\delta\theta \simeq \frac{1+\gamma}{2}\,1''.75 \tag{1.68}$$

The difficulties of the optical lightstar experiments to be performed during a solar eclipse have to do with weather conditions and variable scale changes between eclipse- and comparison- field exposures.

Recently, things have dramatically changed, thanks to very-long-baseline radio interferometry (VLBI, see for instance [Lebach et al. 1995]). This technique proves able to measure angular separations and changes in angles as small as 10^{-4} seconds of arc. When groups of strong quasi-stellar radio sources (quasars) pass very close to the Sun, the angular separations between pairs of them varies. Such variations can be predicted very carefully by studying the motion of the Earth with respect to the Sun, and observations can be used to fit the coefficient $(1+\gamma)/2$ in (1.67). Deflection experiments' results overcame Viking's ones first with [Lebach et al. 1995] which got $(1+\gamma)/2 = 0.9998 \pm 0.0008$. A recent analysis over 2 millions VLBI observations has yielded [Eubanks et al. 1999]

$$\frac{1+\gamma}{2} = 0.99992 \pm 0.00014\,. \tag{1.69}$$

(or $\gamma = 0.9996 \pm 0.0017$) improving the precision of roughly a order of magnitude. This gives a constraint on α through equation (1.57):

$$\alpha^2 \leq 10^{-4}. \tag{1.70}$$

1.2.4 Testing SEP and the parameter β

The next experiment that we consider is in fact a direct test of the strong equivalence principle (SEP), and is based on the observation of Earth and Moon's trajectories as they fall in the gravitational field of the Sun. Since the two planets have non-negligible gravitational binding energies, they could fall with different acceleration toward the Sun, thereby violating WEP for gravitating bodies and then SEP. The most appropriate way to see this and to quantify such violations in terms of the parameters γ and β is to calculate at 1PN order the interaction lagrangian between two massive bodies, just as we did at Newtonian order in (1.28). This turns out to be of the form [Damour and Esposito-Farèse, 1992]

$$L^{\text{int}} = \frac{GM_1M_2}{r}\left[1 + (4\beta - \gamma - 3)\left(\frac{E_1^{\text{grav}}}{M_1\,c^2} + \frac{E_2^{\text{grav}}}{M_2\,c^2}\right) + \text{vel. dependent terms}\right],$$

where all quantities are defined as in (1.28), G is defined in (1.29) and E^{grav} is a suitably defined gravitational self-energy of the body. The combination of parameters $\eta \equiv 4\beta - \gamma - 3$, which in GR adds up to zero, is also called the Nordtvedt parameter. It tells how much the dynamics of the two bodies, and thus their trajectories, depend on their gravitational self energies.

An intuitive way to see this effect is the following. Even if the scalar field φ has a fixed background (cosmological) value, it fluctuates in the presence of a self-gravitating system. In fact, at Newtonian order, it must satisfy equation (1.19) having the trace of the stress-energy tensor as a source and (1.31) as solution:

$$\varphi \;=\; -\,\alpha\,\frac{U_{\text{New}}}{c^2} \;=\; -\,\frac{\alpha\,\widetilde{U}_{\text{New}}}{c^2(1+\alpha^2)} \qquad (1.71)$$

In the last equality (1.53) has been used to express φ in terms of the physical Newtonian-order potential $\widetilde{U}_{\text{New}}$ and we recall that, by convention, φ already indicates the shift of the scalar field from its background value φ_0. Since in this scalar tensor framework the effective Newton constant G depends on φ according to (1.29) a self-gravitating body generally "feels" a modified value of G in its neighborhood. The point is that the total mass of the body also depends on G through its gravitational self-energy E_{grav}, say

$$m^{\text{tot}}c^2 \;=\; mc^2 + E_{\text{grav}}\,. \qquad (1.72)$$

In fact, $E_{\text{grav}} = G \times$ *something*, where, as in the electrostatic case, "something" will be an appropriate double integral adding up the potential energies between each pairs of particles in the body. Here it is sufficient to note that

$$E_{\text{grav}} \;=\; \frac{c^2\,\partial\, m^{\text{tot}}}{\partial \ln G}\,. \qquad (1.73)$$

The equation of motions for a particle of variable mass in a curved space-time is

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds} \;=\; -\left[g^{\mu\nu} + \frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right]\frac{\partial \ln m^{\text{tot}}}{\partial x^\nu}, \qquad (1.74)$$

the geodesic equation being recovered in the limit of constant mass. For a particle initially at rest ($dx^i/ds = 0$), the three-acceleration gets a correction

$$\delta \mathbf{a} \;=\; -\,c^2\,\nabla \ln m^{\rm tot} \;=\; -\,c^2\,\frac{\partial \ln m^{\rm tot}}{\partial \ln G}\,\frac{\partial \ln G}{\partial \varphi}\,\nabla \varphi\,. \tag{1.75}$$

with respect to the usual geodesic one. By using for each factor on the RHS of (1.75) equations (1.73), (1.29) and (1.71) respectively we finally obtain

$$\delta \mathbf{a} \;=\; \frac{E_{\rm grav}}{mc^2}\,(4\beta - \gamma - 3)\,\nabla \widetilde{U}_{\rm New}\,. \tag{1.76}$$

Again, in General Relativity the trajectory of a body is independent of its gravitational self-energy $E_{\rm grav}$, while in a general scalar tensor framework this is no longer true and the Nordtvedt parameter $\eta = 4\beta - \gamma - 3$ gives the strength of such SEP violation. This effect was discovered by [Nordtvedt, 1968] who used a early form of PPN formalism and derived the consequences of such effect on the Earth-Moon orbit, such as the polarization of the Moon's orbit around the Earth toward the Sun.

In 1969 the Apollo 11 mission left a panel of corner-cube laser reflectors on the surface of the Moon. Within a few weeks the telescope at the McDonald Observatory on Mt. Locke, Texas, succeeded in detecting photons returned from a laser pulse sent to the reflector. Two more reflectors were left by Apollo 14 and 15 and a last one in 1973 by the Russian spacecraft Lunakhod II. Since then, lunar laser-ranging (LLR) experiments have made regular measurements of the round-trip travel times of laser pulses between a network of observatories and these lunar retroreflectors, with accuracies that approach 50 ps (1 cm). A recent analysis of 24 years of experimental data has given [Williams, Newhall and Dickey, 1996]

$$\eta \;\equiv\; 4\beta - \gamma - 3 \;=\; (0.7 \pm 1) \times 10^{-3} \tag{1.77}$$

that, combined with (1.69), gives

$$\beta - 1 \;=\; -\,(0.3 \pm 0.5) \times 10^{-3}. \tag{1.78}$$

It should be noted that the above results are valid under the assumption that there is no WEP violations in the Earth-Moon system, as in the case of the metric couplings that we have been considering in this section. To introduce the argument of the next section one may note that an effect associated with the *different compositions* of the Earth and the Moon may also contribute to the acceleration of a self-gravitating body, say,

$$\delta \mathbf{a} \;=\; \delta \mathbf{a}_{\rm SEP} \,+\, \delta \mathbf{a}_{\rm CD} \tag{1.79}$$

where the *composition dependent* contribution $\delta \mathbf{a}_{\rm CD}$ to the acceleration can be defined as in (1.75) apart from considering the variations of the mass due to the specific composition of the body and not to its gravitational self-energy.

At the time when [Williams, Newhall and Dickey, 1996] found their results the estimated uncertainties related to WEP violations in the Earth-Moon system were a factor of 5 greater than those related to the LLR data. Thus, by including the possibility of WEP violations, such uncertainties were inherited by the parameters η and β above. Recently, improving the method of [Su *et sl.,* 1994], [Baessler *et al.*, 1999] removed this ambiguity in the LLR test: they constructed test bodies with compositions very close to that of the actual Earth and Moon but, of course, with negligeable self-gravity. Then, by using a torsion balance, they compared the accelerations of these "miniature" earth and "miniature" moon toward the sun, thereby isolating the composition dependent effect on the Earth-Moon system. The results they obtain have the same confidence level as those of LLR and are valid also in the case of WEP violations. For the Nordtvedt parameter $\eta \equiv 4\beta - \gamma - 3$ they obtain

$$|\,\eta\,| \;\leq\; 1.3 \times 10^{-3} \tag{1.80}$$

that, combined with (1.69), gives

$$|\,\beta - 1\,| \;=\; 5 \times 10^{-4}. \tag{1.81}$$

To summarize, the violations of the *strong* equivalence principle are constrained by the bounds (1.69) [or (1.70)] and (1.81) on the Post-Newtonian parameters γ (or α) and β respectively.

1.3 Non-metric couplings and composition-dependent EP violations

If the scalar theory in (1.1) is *non-metrically coupled*, i.e. if (1.5) does not hold, composition dependent violations of the equivalence principle are generally expected. An intrinsically non-metric coupling is, for instance, that of a gauge theory in 4 dimensions with a ϕ-dependent coupling $[B_F(\varphi) \equiv g^{-2}]$:

$$S_g[A^a_\mu,\; g_{\mu\nu},\; \varphi] \;=\; -\,\frac{1}{4}\int d^4x\sqrt{-g}\,B_F(\varphi)\,F^a_{\mu\rho}F^a_{\nu\sigma}g^{\mu\nu}g^{\rho\sigma}\,. \tag{1.82}$$

In fact, with the rules given in Appendix A one may verify that the combination $d^4x\sqrt{-g}\,g^{\mu\nu}\,g^{\rho\sigma}$ is invariant under Weil rescaling.

Another example is that of a theory where different particles' masses have different dependences on ϕ. The two examples are in fact interrelated: The atomic masses get different QED contributions according to proton and neutron numbers. As a consequence, in models where the electron charge e ($= g$ for QED) has a φ dependence as in (1.82), one generally expects a *non-universal* φ-dependence of particles' masses. In the next subsection we see how the latter occurrence leads to WEP violations. Here it is worth saying that, also because of the above interrelation, the cases of pure metric couplings with SEP violations but without WEP violations are somehow artificial. In all unifying frameworks *both* effects are predicted at some level and then need to be tested separately. In [Damour, 1996*a*]'s words "[...] any bias towards preferentially testing the class of so-called metric theories of gravitation [...] is quite unjustified, both from a historical perspective and (which is most important) from the point of view of the current overall framework of fundamental physics. [...] I know of no cases where an exact metric coupling appeared naturally."

1.3.1 Different particles, different masses (' φ-dependence)

The effective theory for barions + scalar-tensor gravity can be described in the Einstein frame by an action of the form

$$S = S_g + S_m = \frac{1}{16\pi G_*}\int d^4x\sqrt{-g}\,[R - 2\,g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi] \\ - \sum_I \int d^4x\sqrt{-g}\,\overline{N}_I(i\not{D} + m_I(\varphi))N_I\,, \quad (1.83)$$

where the Einstein-frame masses of the different fields N_J generally depend on φ in different ways, more precisely, $d\log m_I/d\varphi \neq d\log m_J/d\varphi$. This is clearly a non-metric coupling for matter. In fact, by the conformal transformation (1.7), Jordan frame squared lengths get a factor $A^2(\varphi)$ and Jordan frame masses get a factor $A^{-1}(\varphi)$ with respect to the Einstein frame ones. So the species N_I is coupled with the "*I-frame*" $\bar{g}^I_{\mu\nu}$, i.e. the frame with respect to which the mass of the particle is φ independent, say,

$$\bar{g}^I_{\mu\nu} = A_I^2(\varphi)g_{\mu\nu}, \qquad \text{with} \qquad \frac{d\log A_I}{d\varphi} = \frac{d\log m_I}{d\varphi}\,.$$

This "I-metric" $\bar{g}^I_{\mu\nu}$ is different from the one that couples to the species J, and a common Jordan frame doesn't exist in this case. By referring to Einstein frame quantities we can see the effects of a varying mass m_I on a particle's trajectory by adapting equation (1.75) as follows

$$\delta\mathbf{a} = -\,c^2\,\nabla\ln m_I(\varphi) = -\,\alpha_I\nabla\varphi \qquad (1.84)$$

where we have introduced the crucial parameter

$$\alpha_I \equiv \frac{d\ln m_I(\varphi)}{d\varphi} \qquad (1.85)$$

that tells the strength of the scalar coupling to I particles. Note that, for the barions + gravity theory (1.83), if all masses have the same φ-dependence then we go back to the metric coupling case of last section and all α_Is in (1.85) coincide with the parameter α defined in (1.12). As usual, the above derivative has to be taken at the present (cosmologically determined) value of φ.

Now we can consider the interaction between two particles I and J by assuming that the scalar field in (1.84) has the particle J as a source. Then we can use the Newtonian order solution (1.31) which gives, for a pointlike source of Einstein frame mass M,

$$\varphi = -\,\alpha_J\,\frac{G_*\,M}{r}\,. \qquad (1.86)$$

Note that, for the field generated by J, the appropriate coupling α_J has to be used. Putting all together, the Einstein frame acceleration of a particle I toward particle J ammounts to

$$|\,\mathbf{a}\,| = \frac{G_*\,M}{r^2}(1 + \alpha_I\,\alpha_J), \qquad (1.87)$$

corresponding to an effective Newton constant of the form.

$$G^E_{IJ} \; = \; G_*(1+\alpha_I\,\alpha_J). \tag{1.88}$$

This simple Newtonian order result generalizes (1.29) to composition dependent couplings. Since there is not a preferred "physical frame" we are referring to Einstein frame quantities.

A crucial observable quantity is the relative difference of acceleration between two test masses, made respectively of I and J particles, falling in the same gravitational field: the so-called "Eötvös ratio" defined as

$$\left(\frac{\Delta a}{a}\right)_{IJ} \;\equiv\; \frac{2\,|\mathbf{a}_I - \mathbf{a}_J|}{|\mathbf{a}_I + \mathbf{a}_J|}\,. \tag{1.89}$$

This is a manifestly frame invariant quantity. At Newtonian order, by (1.87) it reads

$$\left(\frac{\Delta a}{a}\right)_{IJ} \;=\; \alpha_E\,(\alpha_I - \alpha_J). \tag{1.90}$$

where α_E refers to the coupling of the common attractor, say, the Earth.

1.3.2 The effect of variable gauge couplings

Now we want to study in a more quantitative way the already mentioned relation between the φ-dependence of the gauge couplings [equation (1.82)] and the φ-dependence of the masses discussed in the last subsection. For this purpose we need to estimate the various gauge contributions to the mass of an atom. We follow a symplified path and neglect from the beginning the contributions of quarks' and electrons' masses. We thus consider only the major contribution coming from pure QCD effects and the QED corrections:

$$m \; = m_{\rm QCD} \; + \; \Delta m. \tag{1.91}$$

The electromagnetic binding energy Δm accounts both for the separate contributions of each nucleon and for a collective Coulomb binding energy term. The latter, which has been argued by [Damour and Polyakov, 1994] to dominate over the others, is proportional to the atomic parameter E, that depends on the number of protons Z and neutrons N through

$$E \;=\; \frac{Z(Z-1)}{(N+Z)^{1/3}}\,. \tag{1.92}$$

At first order in the fine-structure constant α_e [not to be confused with dilaton couplings (1.12) and (1.85) !] we thus have

$$\Delta m \;\simeq\; a_3\,u_3\,\alpha_e E \tag{1.93}$$

where the proportionality factor has been written as a pure number $a_3 \simeq 0.105$ times the atomic mass unit $u_3 \simeq 931.5$ MeV.

The coupling α_I of the dilaton to an element I is then given, according to (1.85), by

$$\alpha_I \simeq \frac{d}{d\varphi} \log \left[m_{\rm QCD} \left(1 + \frac{\Delta m_I}{m_{\rm QCD}} \right) \right] \simeq \alpha_{\rm had} + a_3 \frac{d\alpha_e}{d\varphi} \frac{E_I}{\mu_I} , \tag{1.94}$$

where μ_I denotes the mass of the element in atomic mass units, $\mu_I = m_I/u_3$, and $\alpha_{\rm had} \equiv d \log m_{\rm QCD}/d\varphi$ is the dominant QCD contribution to α. The dilaton dependence of $m_{\rm QCD}$ is encoded in $\Lambda_{\rm QCD}$, the Einstein frame confinement QCD scale. The relevant QCD part of hadron masses is in fact known to be proportional to $\Lambda_{\rm QCD}$ with some pure number as proportionality constant. We have

$$\alpha_{\rm had} = \frac{d \log \Lambda_{\rm QCD}(\varphi)}{d\varphi} \tag{1.95}$$

The parameter $\alpha_{\rm had}$ clearly measures a *composition independent* effect and, in this non metric-coupling model, is the best candidate to replace the α parameter defined in (1.12) in the context of exact metric coupling theories. As such, it also has to satisfy the experimental constraint (1.70) in order for the theory to be acceptable.

If we trust (1.82) as a tree-level theory at some fundamental scale Λ_s we can infer the dilaton dependence of $\Lambda_{\rm QCD}$ on the basis of renormalization group arguments. It is worth saying that since Λ_s has the dimensions of a mass we have to specify the frame where it is fixed (φ-independent). In string inspired models such frame is the *string frame* [the one of equation (1.1)] and Λ_s is the string mass. Here, for semplicity, we assume the scale Λ_s to be fixed once and for all in the Einstein frame, and refer to Section 3.2 for a more detailed calculation that takes into account the (slight) E-frame φ-dependence of Λ_s. At one loop order in perturbation theory the coupling g_μ^2 at some scale μ is given with respect to a reference scale Λ_s by [Peskin and Schroeder, 1995]

$$\frac{1}{g_\mu^2} = \frac{1}{g_{\Lambda_s}^2} + \frac{1}{\beta} \log \frac{\mu}{\Lambda_s} , \tag{1.96}$$

where $g_{\Lambda_s}^2$ is supposed to be much less than unity for perturbation theory to be trusted and β is some calculable parameter of order one. The confinement scale $\Lambda_{\rm QCD}$, where perturbation theory breaks down $g_{\Lambda_{\rm QCD}}^2 \simeq 1$, corresponds to

$$\Lambda_{\rm QCD} \simeq \Lambda_s \exp(-\beta/g_{\Lambda_s}^2) . \tag{1.97}$$

Then $\Lambda_{\rm QCD}$ inherits the φ dependence from $B_F(\varphi) \equiv g_{\Lambda_s}^{-2}$ and we finally obtain

$$\alpha_{\rm had} \simeq \ln \frac{\Lambda_s}{\Lambda_{\rm QCD}} \frac{d \log B_F^{-1}}{d\varphi}. \tag{1.98}$$

Enough for composition independent-QCD effects. Now let's come back to the composition independent ones where the residual QED effects enter through the fine structure constant $\alpha_e(\varphi)$. The "Eötvös ratio", is obtained by substituting (1.94) into (1.90):

$$\left(\frac{\Delta a}{a}\right)_{IJ} \simeq \alpha_{\rm had}\, a_3\, \frac{d\alpha_e}{d\varphi}\, \Delta\left(\frac{E}{\mu}\right)_{IJ}, \tag{1.99}$$

where, for any quantity A, $\Delta(A)_{IJ} \equiv A_I - A_J$. A table with the values of E/μ for several elements can be found in [Damour, 1996*b*].

1.3.3 Testing WEP

The first (W)EP tests date back to Galileo's Leaning Tower of Pisa experiments, whose precision was about 1%. Newton improved the experimental accuracy to 0.1% using pendula made of different materials. Since then, accuracies have dramatically improved. Modern high-precision experiments are mostly of the "Eötvös type" named after Lorànd Eötvös who started to experiment with gravity and the torsion balance around 1885. In brief, two objects of different composition are placed on a tray and suspended in a horizontal orientation by a fine wire ("torsion balance"). The entire apparatus is continuously rotating around a vertical axis with a period τ long compared to the torsional oscillation period of the pendulum. The measured effect is the modulation (with period τ) in the torque on the two objects, expected in the case of WEP violations.

Recently, the "Eöt-Wash" experiment [Su *et sl.,* 1994] carried out at the University of Washington used a very sophisticated version of torsion balance and compared the acceleration of various materials toward local topography on the Earth, movable laboratory masses, the Sun and the galaxy. They measured the differential acceleration between Be-Cu and Be-Al test body pairs [the ratios E/μ in equation (1.99) for aluminium, beryllium and copper are respectively 1.93, 0.64, 3.2], obtaining

$$\begin{aligned}\left(\frac{\Delta a}{a}\right)_{\rm Be-Cu} &= (-1.9 \pm 2.5) \times 10^{-12} \\ \left(\frac{\Delta a}{a}\right)_{\rm Be-Al} &= (-0.2 \pm 2.8) \times 10^{-12}\end{aligned} \tag{1.100}$$

Curiously enough, the next generation of WEP tests [Blazer, 2001] will be an improved version of Galileo's Leaning Tower of Pisa experiment, in that they will measure the differential acceleration of two truly freely falling bodies. Such experiments will be made on satellites so that they will not have to deal with environmental instabilities induced by Earth's density fluctuations and by human activities. Moreover, they will be able to obtain a very long duration of free fall, following the test bodies along several orbits around the Earth. The approved Center National d'Etudes Spatiales (CNES) mission MICROSCOPE [P. Touboul *et al.*, 2001] will fly in 2004 and will explore the level $\Delta a/a \sim 10^{-15}$, while the planned National Aeronautics and Space Agency (NASA) and European Space Agency (ESA) mission STEP (Satellite Test of the Equivalence Principle)[P. W. Worden, 1996], [STEP] could explore the $\Delta a/a \sim 10^{-18}$ level.

Chapter 2

Gravity and Strings

In this chapter we give a brief introductional review of the low energy limit of superstring theory, focussing on its implications on gravity. We first recall those aspects which seem at odds with phenomenology and then review the strategies by which these problems have been addressed up to date. The last section of this chapter is devoted to describe the "strong coupling scenario" which is the subject of this thesis work and which may suggest a new framework to interpret string theory in a phenomenologically acceptable way.

2.1 The tree-level action

The low-energy limit of superstring theory [Green, Schwarz and Witten, 1987], [Polchinski, 1998] is a supersymmetric quantum field theory of massless particles. Such a field theory should be viewed as an effective UV cut off theory, the cut off being given by the string scale M_s, that we take as follows

$$\frac{M_s^2}{2} = \frac{1}{\alpha'} = 2\pi T\,, \tag{2.1}$$

where α' is the universal Regge slope parameter[1] with dimensions of $(\text{length})^2$ and T is the string tension. The only other parameter of the theory is the dimensionless string coupling g_s, which is given by the vacuum expectation value (VEV) of a scalar field, the *dilaton* Φ

$$e^{\langle\Phi\rangle} = g_s^2\,. \tag{2.2}$$

The so-called critical superstring, is defined in $D = 10$ space-time dimensions. There are various types of string theories: type I, heterotic $E_8 \times E_8$ and heterotic $SO(32)$ that have $\mathcal{N} = 1$ supersymmetry. Type II A and II B which have $\mathcal{N} = 2$. Heterotic theories and type I are the only theories with a non-abelian gauge field.

[1] We apologize with the reader for the omnipresent use of the greek letter α which indicates the strength of the dilaton's coupling to matter [equations (1.12) and (1.85)], the gauge couplings as the fine structure constant in (1.99) and now, with a prime, the Regge slope parameter.

The low energy effective action of *heterotic* strings looks like

$$S = \int d^{10}x\sqrt{-G}e^{-\Phi}\left[\frac{1}{\alpha'^4}(R_{10} + \partial_\mu\Phi\partial^\mu\Phi) - \frac{1}{4\alpha'^3}\,\mathrm{Tr}\,F^2 + \dots\right]. \qquad (2.3)$$

We are concentrating on the gauge and gravitational sectors of the theory and ellipsis stand for the other massless fields coming out from string quantization. R_{10} is the Ricci scalar for the 10-dimensional metric tensor $G_{\mu\nu}$, whose appearance is one of strings' "miracles", and F is the gauge field strength, the gauge group for heterotic string being either $E_8 \times E_8$ or $SO(32)$.

Compactification over a 6-dimensional Calabi-Yau manifold of volume V allows to

- interpret the theory as effectively 4-dimensional on length scales $\gtrsim V^{1/6}$,
- break supersymmetry,
- break the gauge group into the Standard Model group $U(1) \times SU(2) \times SU(3)$.

Upon compactification, the internal volume V becomes a new field and the sector of the theory relevant for this discussion can be described by a 4-dimensional action similar in form to (2.3) where the *four dimensional dilaton*

$$\phi \equiv \Phi - \log\frac{V}{\alpha'^3}, \qquad e^{\phi} \equiv \frac{e^{\Phi}\,\alpha'^3}{V} \qquad (2.4)$$

takes the place of Φ and the various fields are effective four dimensional quantities:

$$S = \int d^4x\sqrt{-\widetilde{g}}\; e^{-\phi}\left[\frac{\widetilde{R}}{\alpha'} + \frac{(\widetilde{\nabla}\phi)^2}{\alpha'} - \frac{1}{4}F^{a\,\mu\nu}F^a_{\mu\nu}\right] + \dots \qquad (2.5)$$

Here ellipsis stand also for terms with moduli fields such as V created by compactification as well as higher derivative operators representing the low-energy effects of all the massive string modes. $\widetilde{R}$ is the Ricci scalar of the 4-dimensional metric $\widetilde{g}_{\mu\nu}$. Tildes emphazise that quantities belong to the *string frame* (see the discussion of Section 1.1).

2.1.1 Energy scales, couplings and the Planck scale problem

It is now essential to identify some scales of phenomenological relevance. We first note that if we want to break the gauge group through compactification then we expect the compactification scale V to be related to the Grand-Unification Scale M_{GUT}. In the context of the Minimal Supersymmetric Standard Model, which successfully incorporates the Standard Model within the framework of a supersymmetric theory, the low-energy couplings extrapolated at high energies meet each other at the scale

$$M_{GUT} \simeq 2 \times 10^{16}\,\mathrm{GeV} \qquad (2.6)$$

where their value is [2]

$$\alpha_{GUT} \simeq \frac{1}{25}. \tag{2.7}$$

In a more or less isotropic Calabi-Yau manifold we then expect

$$V \simeq M_{GUT}^{-6} \tag{2.8}$$

which fixes the scale of the compactification volume on phenomenological grounds.

Now a few words on the crucial role of the dilaton ϕ in the theory. Its VEV relates to the tree-level couplings of gauge and gravitational interactions in four dimensions, and tells the degree of confidence by which the very tree-level action (2.5) can be trusted. The physics of the string vacua with $e^{\langle\phi\rangle} > 1$ may be as apparently far from (2.5) as low energy, hadronic physics is from the perturbative QCD lagrangian. The same is true for the 10-dimensional theory (2.3): the tree-level string coupling g_s defined in (2.2) gives the confidence level of perturbative heterotic string theory. In this respect, the discovery of dualities [Polchinski, 1998] during the 90s between different string theories has given some insight into the strong coupling regime and the belief that the different string theories are in fact different limits of a unique theory. Just to cite some results concerning heterotic strings, the strong coupling limits of $SO(32)$ and $E_8 \times E_8$ have been proved to be equivalent to the weak coupling limits of *type I* (open) string and 11-dimensional M-theory compactified on $R^{10} \times S^1/Z_2$ respectively.

The case $e^{\langle\Phi\rangle} < 1$, $e^{\langle\phi\rangle} < 1$

If both the four dimensional and the ten dimensional theories are not strongly coupled the couplings can be directly read off (2.5):

$$\frac{M_P^2}{2} \simeq \frac{e^{-\phi}}{\alpha'} = \frac{e^{-\Phi}V}{\alpha'^4} \tag{2.9}$$

$$\alpha_{GUT} \simeq \frac{e^{\phi}}{4\pi} = \frac{e^{\Phi}\alpha'^3}{4\pi V}, \tag{2.10}$$

where $M_P \equiv (8\pi G)^{-1/2}$ is the reduced Planck mass. Note that the universal coupling of the dilaton to both the gravitational and the gauge fields typical of heterotic string theory forces a relation between Planck and string scales:

$$M_P^2 = \frac{M_s^2}{4\pi\alpha_{GUT}}. \tag{2.11}$$

By substituting the expected volume of compactification (2.8) into (2.9) with α' given by (2.10) we get :

$$\frac{M_P}{M_{GUT}} \simeq (4\pi\alpha_{GUT})^{-2/3}\, e^{\Phi/6} \tag{2.12}$$

[2]For a nice review on the argument and its relations to string theory see [Dienes, 1997].

this is clearly too low a value for the Planck mass over GUT scale ratio, since under the non-strong coupling assumption $e^{\Phi} < 1$ and by substituting (2.7) we would get the inequality

$$\frac{M_P}{M_{GUT}} \overset{?}{<} 1.6\,. \tag{2.13}$$

This perturbative heterotic string prediction is clearly in contrast with the measured value of the reduced Planck mass

$$M_P \simeq 2.4 \times 10^{18}\,\mathrm{GeV}\,, \tag{2.14}$$

which gives

$$\frac{M_P}{M_{GUT}} \simeq 120. \tag{2.15}$$

According to [Kaplunovsky, 1988] the one loop corrections to the tree level Planck mass (2.11) improve the bound (2.12) by a order of magnitude but this is still incompatible with the observed value (2.14) .

The case $e^{\langle\Phi\rangle} > 1$, $e^{\langle\phi\rangle} < 1$

Using duality arguments, [Witten, 1996] suggested that the solution to the above problem may be addressed by going to strong coupling while keeping the effective four dimensional coupling weak ($e^{\Phi} > 1$, $e^{\phi} < 1$). This works both for $E_8 \times E_8$ and for $SO(32)$ heterotic string theory.

The case of $SO(32)$ is particularly simple since the heterotic theory at *strong coupling* is equivalent to the *weakly coupled* Type I theory, which is known to be able to provide a much better agreement between M_P and M_{GUT}. In Type I theory, in fact, the dilaton does not couple universally to gauge and gravitational fields and in the effective 10 dimensional action the F^2 term is multiplied by $e^{-\Phi/2}$ instead of the overall factor $e^{-\Phi}$ that appears in (2.3):

$$S_{\mathrm{type\,I}} = \int d^{10}x\sqrt{-G_I}\left[\frac{e^{-\Phi_I}}{\alpha'^4}(R_{I\,10} + \partial_\mu\Phi_I\partial^\mu\Phi_I) - \frac{e^{-\Phi_I/2}}{4\alpha'^3}\,\mathrm{Tr}\,F^2 + \ldots\right],$$

where the subscript I indicates the quantities relative to type I theory which are related to the heterotic ones by the duality transformations [Polchinski, 1998]

$$G_{I\,\mu\nu} = e^{-\Phi/2}\,G_{\mu\nu}, \quad \Phi_I = -\Phi, \quad V_I = e^{-3\Phi/2}V \tag{2.16}$$

where the last equality relates the compactification volumes in the two theories. In brief, instead of (2.10), perturbative type I theory provides

$$\alpha_{GUT} \simeq \frac{e^{\Phi_I/2}\,\alpha'^3}{4\pi V_I}\,, \tag{2.17}$$

and instead of (2.12) we have

$$\frac{M_P}{M_{GUT}} \simeq (4\pi\alpha_{GUT})^{-2/3}\, e^{-\Phi_I/6}\,, \tag{2.18}$$

which is not bounded from above but, rather, can take arbitrarily large values by going deep in the weak coupling region of type I theory $\Phi_I \to -\infty$ i.e. the *strong coupling* region for heterotic $SO(32)$! What about the heterotic four dimensional coupling? By using (2.4) and the transformations (2.16) one can see that still α_{GUT} defined in (2.17) is given by $\simeq e^{\phi}/4\pi$ which thus must be a smallish number. As announced, we are strongly coupled in ten dimensions but still perturbative in four.

In the very same spirit but with a much more technical argument, [Witten, 1996] argues that $E_8 \times E_8$ heterotic string theory at strong coupling provides the estimate

$$\frac{M_P}{M_{GUT}} \simeq \alpha_{GUT}^{-1}, \tag{2.19}$$

which is better than the tree-level weak coupling estimate (2.12).

2.1.2 String and gravity at tree-level

Some general features of the four dimensional effective action (2.5) are shared by many types of string theories/compactification mechanisms at tree level as well as by several supergravity models. It has been noted in the last section that in the effective theory (2.5) couplings are not *a priori* given but, rather, dynamical quantities related to the VEV of the dilaton. Although this is a welcome occurrence for a theory candidate for unifying all the interactions (we don't need to fix the value of any free parameter by hand), as we have stressed in Chapter 1, we are provided with strict phenomenological bounds on the possible variations of the coupling constants, most notably of the fine structre constant. Related to this, the presence of coupled massless scalar fields generally leads to equivalence principle violations. Needless to say, a theory like that in (2.5) describes something profoundly different from gravitation as we experience it. Although we have in general little knowledge of the matter content of realistic string models, we argue from (2.5) that both QED and QCD couplings have an exponentially ϕ-dependence. As emphasized in Section 1.3, such a dependence cannot be cancelled out by a conformal frame transformation (see Section 1.1 for details) and represents an intrinsecally non-metric coupling. From action (2.5) we then estimate the derivative of the fine structure constant α_e to be of the same order of α_e itself,

$$\frac{d\alpha_e(\varphi)}{d\varphi} \simeq \alpha_e \tag{2.20}$$

[the canonical field (1.3) is just ϕ rescaled by a costant factor] while from (1.98) we get

$$\alpha_{\text{had}} \simeq \frac{M_s}{\Lambda_{\text{QCD}}} \simeq 40 \tag{2.21}$$

which is unacceptable in virtue of the limit (1.70). The order of magnitude of a typical Eötvös ratio (1.99) is also dramatically incompatible with the bound (1.100) since it gives

$$\left(\frac{\Delta a}{a}\right) \simeq 3 \times 10^{-2}. \tag{2.22}$$

In the rest of this chapter we examine the main different theoretical schemes for obtaining a phenomenologically acceptable gravity from string theory. In the first section we consider the conventional solution to the above problems, according to which the dilaton, as well as all moduli fields, acquires a mass for some non-perturbative mechanism. Next, we review the Damour-Polyakov "least-coupling" principle, of which the *strong coupling scenario*, may be considered as an extension. The basic theoretical motivations of the strong coupling scenario are analised in Section 2.3, while its phenomenology is discussed in detail in the next chapter.

2.2 How to get a theory phenomenologically ok?

In the last section we have faced two problems, the first is related to the expected ratio M_P/M_{GUT} in string theory, the second (also called "*moduli stabilization problem*") relates to equivalence principle violations as well as temporal variations of the couplings. In subsection 2.1.1 we have seen that the first problem can be eased by going to strong string coupling while keeping the 4-dimensional couplings small ($e^{\langle\Phi\rangle} > 1$, $e^{\langle\phi\rangle} < 1$). In order to address the moduli stabilization problem in the perturbative region of the theory (see Fig. 2.1), one may ask if loop corrections improve the bad tree-level estimates (2.20)-(2.22). The answer is *no*: [Taylor and Veneziano, 1988] have shown that the one-loop corrected couplings of matter to the dilaton remain as strong as those of gravity. It is then conventional wisdom to assume the existence of a potential $V(\phi)$ which both fixes the dilaton value at its minimum (thereby preventing unacceptable variations of coupling constants) and gives it a mass (Figure 2.1). The interactions of a scalar field of mass m_ϕ are in fact suppressed over distances $l > 1/m_\phi$ and give a Yukawa-type contribution to the gravitational interaction between two bodies. This has been derived in some detail in Section 1.2 and expressed by (1.28) . With a dilaton of mass, say, $m_\phi \gtrsim 10^{-3}\,\mathrm{eV}$ deviations from Einstein's gravity would be quenched on distances larger than a fraction of a millimeter and then unobservable in current experiments.

A dilaton stabilization at weak coupling, however, looks improbable for theoretical reasons. At weak coupling, in fact, we have to trust (2.5), and a potential for the dilaton does not appear in this action. In other words, it seems improbable that a non-perturbative potential may act at weak coupling where it should rapidly fall to zero.

2.2.1 Damour, Polyakov and the "least coupling principle"

The only alternative left to consider seem to be the non-weak 4-dimensional regime: $e^\phi \gtrsim 1$. Although at first sight inconsistent [by (2.5) we would get,

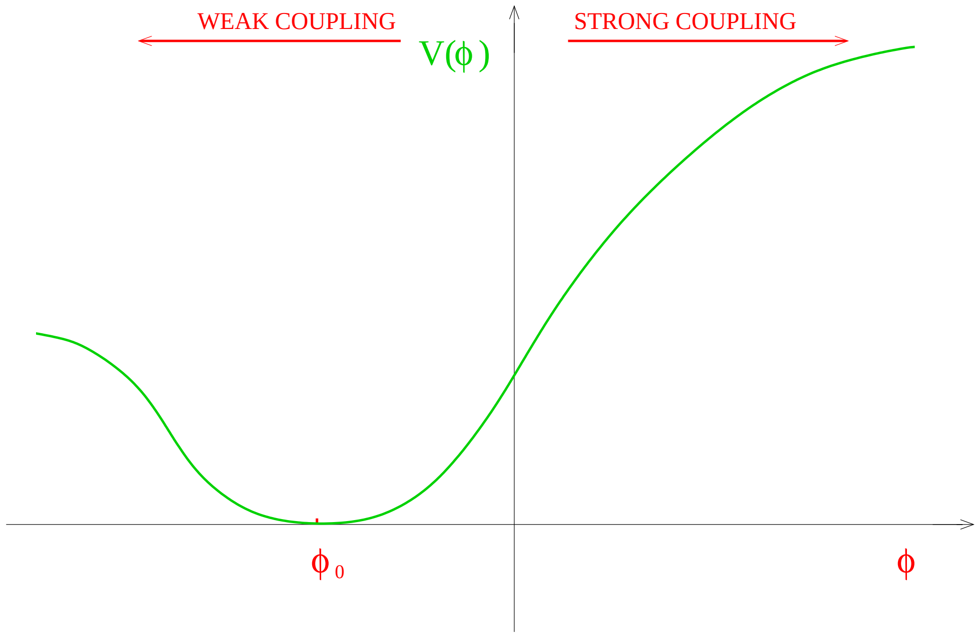

Figure 2.1: The different regimes of the 4-dimensional effective theory depend on the VEV of the four dimensional dilaton ϕ. The vertical axis at $\phi = 0$ separates weak coupling from strong coupling and a potential $V(\phi)$ for the dilaton is drawn. At one time, $V(\phi)$ freezes the dilaton value in the perturbative region and gives it a mass, thereby suppressing its long-range interactions.

for instance, $\alpha_{GUT} \simeq 1$], this alternative deserves to be considered since, as already stressed in Section 2.1.1, as one moves from the weak coupling region, loop corrections become more and more important, and one should trade the tree-level action (2.5) for an *effective action* of the generic form

$$\Gamma = \int d^4x \sqrt{\widetilde{g}} \left(\frac{B_g(\phi)}{\alpha'} \widetilde{R} + \frac{B_\phi(\phi)}{\alpha'} [2\widetilde{\Box}\phi - (\widetilde{\nabla}\phi)^2] - \frac{1}{4} B_F(\phi) \widetilde{F}^2 + \cdots \right) . \tag{2.23}$$

where the functions $B(\phi)_i$ will admit a generic weak coupling expansion of the type

$$B_i(\phi) \;=\; e^{-\phi} + c_0^{(i)} + c_1^{(i)} e^{\phi} + c_2^{(i)} e^{2\phi} + \ldots . \tag{2.24}$$

In (2.23) the kinetic term for the dilaton is defined in such a way that B_ϕ has the correct weak coupling expansion (2.24), consistent with (2.5) but does not change sign when going from weak to strong coupling.

[Damour and Polyakov, 1994] has considered a theory of this type and formulated a "*least coupling principle*" that can be stated as follows:

If there exists a special value ϕ_m which extremizes all the (relevant) coupling functions $B_i^{-1}(\phi)$, the cosmological evolution of the system naturally drives ϕ

toward ϕ_m, where the theory is phenomenologically safe.

In the model considered by [Damour and Polyakov, 1994], hence, the dilaton ϕ does not acquire a mass but, through its cosmological evolution, *decouples* from all the other fields including gravity, thereby restoring the equivalence principle.

The crucial assumption, that all the (relevant) coupling functions $B_i^{-1}(\phi)$ extremize at some (and the same) ϕ_m is motivated by the conjectured S-duality of heterotic strings compactified on a 6-dimensional torus T^6 [Sen, 1994]: the invariance of the theory under $\phi \longleftrightarrow -\phi$. Of course another crucial assumption is that the functions B_i in ϕ_m have effectively the values of the respecting coupling constant e.g. $B_F(\phi_m)^{-1} \simeq 4\pi\alpha_{GUT}$.

The details of the Damour-Polyakov model will not be given here. Some of the calculations are similar to those that will be presented in Chapter 3 for the "strong coupling scenario" although the results are different. Contrary to the latter, in fact, the Damour-Polyakov model does not predict any violations of the equivalence principle to be detected in next to come experiments.

2.3 The strong coupling scenario

In this section we introduce the strong coupling scenario, the main subject of this thesis work. We first give the theoretical motivations why the dilaton should effectively decouple from the other fields as in the Damour-Polyakov model and why the extremizing value ϕ_m should be at infinity, $\phi_m = \infty$. The same theoretical motivations suggest also that the functions B_i at infinity have roughly the required values to be effectively interpreted as "couplings", a circumstance that was simply *assumed* in the Damour-Polyakov work. Equivalence principle violations and couplings' time variations of the model are analysed in detail in Chapter 3, while possible cosmological consequences are drawn in Chapter 4. Some other interesting physical implications of the strong coupling scenario are summarized at the end of this section.

2.3.1 Theoretical motivations

In what follows we use field-theoretic arguments in order to figure out the effective action of string theory i.e. the form of the coupling functions $B_i(\phi)$ in equation (2.23). The underlying assumption is reasonable from a stringy point of view: all the low-energy physics is encoded in the 4-dimensional quantum field theory described at the tree-level by the action (2.5).

Following [Veneziano, 2002], we thus calculate the effective action[3] for gauge and gravitational interactions in a toy model endowed with a cut off Λ (which is assumed to preserve gauge symmetry and general covariance just as the "real" string theory cut off M_s) and in the presence of a large number of spin 1/2 and 0 massive fields [which represents the fermionic sector of the theory and the ill-understood scalar sector: Higgs fields, moduli fields, ... both omitted in

[3] See, for instance, [Peskin and Schroeder, 1995]

(2.5)]. Some details of the calculation are given in Appendix B while in what follows we report and discuss the results.

We consider a quantum field theory in D dimensions whose tree-level action has the form

$$S = \int d^D x \sqrt{g} \left[\frac{R}{2\,\kappa_0^2} - \frac{F^2}{4\,g_0^2} \right] + \sum_b^{N_0} S_{\rm scal}(\phi_b) + \sum_d^{N_{1/2}} S_{\rm ferm}(\psi_d)\,,$$

where Λ is the cut-off, κ_0 and g_0 are the bare gravitational and gauge couplings, and $F^a_{\mu\nu}$ is the field strength of the gauge field A^a_μ. For simplicity we assume that the N_0 scalars ϕ_b have the same mass m_0 and the $N_{1/2}$ fermions ψ_d have the same mass $m_{1/2}$. The third "large" number is the rank N_1 of the gauge field. We also assume that they are minimally coupled. Of course, since we consider a generic background, the metric field $g_{\mu\nu}$ and the gauge field A^a_μ also enter the matter sector of the model, $S_{\rm scal}$ and $S_{\rm ferm}$ through covariant derivatives.

In order to describe the effective dynamics of gauge and gravitational interactions we first integrate out the matter fields ϕ_d and ψ_b.

$$e^{iS_{\rm eff}[g_{\mu\nu},A^a_\mu]} = \int \mathcal{D}[\phi_b, \psi_d]\; e^{iS[g_{\mu\nu},\; A^a_\mu,\; \phi_b,\; \psi_d]}\;. \tag{2.25}$$

As a result, new (local as well as non-local) terms for $g_{\mu\nu}$ and A^a_μ are produced. This is not a novelty: in the very same way, for instance, four fermions interactions *à la* Fermi are produced if one integrates out the massive $W^\pm$ and Z^0 gauge bosons of electro-weak interactions. The example of *induced gravity* [Sakharov, 1968] fits even better: even if not present at tree level, a Ricci term in the action is generally produced by quantum effects. Hence gravity may be a "quantum by-product" of the matter fields.

At leading order in the cut-off (and zeroth order in the field derivatives) the integration (2.25) produces a cosmological constant term, which will be assumed to vanish for some mechanism. At second order in the field derivatives, we have corrections to the Ricci and F^2 terms which renormalize the coupling constants as follows:

$$\frac{1}{2\kappa_0^2} \;\longrightarrow\; \frac{1}{2\kappa_0^2} + (c_0 N_0 + c_{1/2} N_{1/2})\Lambda^{D-2} \tag{2.26}$$

$$\frac{1}{4g_0^2} \;\longrightarrow\; \frac{1}{4g_0^2} + (b_0 N_0 + b_{1/2} N_{1/2})\Lambda^{D-2} \tag{2.27}$$

In Appendix B c_0 and $c_{1/2}$ are calculated for the model at hands using results of heat-kernel techniques [Barvinsky and Vilkovisky, 1985], [Avramidi, 1995], while b_0 and $b_{1/2}$ can be obtained with more common quantum field methods [Peskin and Schroeder, 1995]. In $D = 4$ the b coefficients account for the contribution of each "flavour" to the beta function. In general they are proportional to

$$b_i \,\propto\, C_i\, d_i / d_A \tag{2.28}$$

where C_i and d_i are the quadratic Casimir operator and dimensionality for the matter representation i respectively and d_A is the dimensionality of the adjoint representation. The precise numerical values of the b and c coefficients depend of course on the renormalization scheme as well as on the explicit implementation of the cut-off. When $D = 4$ the behaviour of the leading order quantum corrections in (2.27) is *logarithmic* in the cut off and reproduces the known logarithmic scaling with the momenta of the renormalization group equations. Note also that the leading corrections to the coupling constants (2.26), (2.27) do not depend on the bare couplings. As a consequence of having considered *free* scalars and fermions, the results (2.26), (2.27) are non-perturbative and account for all the infinite loops resummation.

We now come to the inclusion of gauge and gravity loops in the effective action and this is where the large-N limit hypothesis helps. In particular, we need that the combination $b_0 N_0 + b_{1/2} N_{1/2} \to \infty$ so that the effective coupling constant after matter-loop renormalization is arbitrarily small. In this case we can limit ourselves to the one-loop contribution to the effective action since it dominates the functional integral. In $D > 4$ the renormalized gauge coupling have the form

$$\frac{1}{4g^2} = \frac{1}{4g_0^2} + (b_0 N_0 + b_{1/2} N_{1/2} + \beta)\Lambda^{D-4} \,. \tag{2.29}$$

The gauge field contribution β is proportional to the quadratic Casimir of the adjoint representation C_A. For a gauge group $SU(N_c)$, thus, $\beta \simeq N_c$, and in terms of the number of gauge bosons N_1 $\beta \simeq N_1^{1/2}$. Again, the case $D = 4$ is more interesting, the dependence on the cut-off is logarithmic and the linear combination in parenthesis adds up to the one loop beta function. More about the $D = 4$ case is found in the original work [Veneziano, 2002].

On the gravity side the situation is simpler and similar for all $D \geq 4$. In the large-N limit graviton loop corrections are subleading and the one-loop gauge field contribution in this case is roughly proportional to the number of gauge bosons N_1 [Adler, 1982]:

$$\frac{1}{2\kappa^2} = \frac{1}{2\kappa_0^2} + (c_0 N_0 + c_{1/2} N_{1/2} + c_1 N_1)\Lambda^{D-2} \,. \tag{2.30}$$

2.3.2 The basic assumptions

Equations (2.29) and (2.30) show that, in a model with a large number of matter fields and a high rank gauge group, the effective gauge and gravitational couplings in cut-off units are bounded by numbers of order N^{-1}, N being the number of independent species. Moreover, the strong bare coupling ("compositeness") limit of the theory, $g_0, \kappa_0 \to \infty$, is well defined and makes sense, being the region of the parameter space where the effective couplings reach their extremum. In the 4-dimensional string theory context we are considering [equation (2.5)], the dilaton relates to the bare tree-level couplings as discussed in detail in Section 2.1.1, and the cut-off Λ is at the string scale M_s. By rereading the results (2.29)–(2.30) with this "string-eye" we infer on the form of the Damour-Polyakov gauge couplings B_i [equation (2.23)] in the region $e^{\langle\phi\rangle} \gtrsim 1$ and state the the basic assumption of the "strong coupling" scenario:

In the effective action of string theory

$$S = \int d^4x \sqrt{\widetilde{g}} \left(\frac{B_g(\phi)}{\alpha'} \widetilde{R} + \frac{B_\phi(\phi)}{\alpha'} [2\widetilde{\Box}\phi - (\widetilde{\nabla}\phi)^2] - \frac{1}{4} B_F(\phi) \widetilde{F}^2 + \cdots \right)$$

all the relevant coupling functions B_i have the general form

$$B_i(\phi) \underset{\phi\to\infty}{=} C_i + \mathcal{O}(e^{-\phi}), \tag{2.31}$$

where C_i are pure numbers of order $\sim 10^2$ (the number of fields we have integrated over).

As anticipated, we assume, as in the Damour and Polyakov model, that all the coupling functions $B_i(\phi)$ have a common extremum, except that such an extremum is at infinity! Thus, the standard scenario of a dilaton *stuck* at a certain value may change in that of a dilaton running to infinity i.e. that of a theory running toward (bare!) strong coupling and still phenomenologically acceptable.

The phenomenological consequences of this new scenario related to the violations of the equivalence principle and to the time variations of the couplings are analyzed in detail in chapter 3, while some cosmological consequences are drawn in chapter 4. In what follows we briefly discuss some other interesting physical implications of this idea.

2.3.3 Possible physical implications

We now go back to the "Planck-scale problem" of Section 2.1.1 and see how it is addressed at 4-dimensional (and 10-dimensional) strong coupling i.e. the region of parameters $e^{\langle\Phi\rangle} > 1$, $e^{\langle\phi\rangle} > 1$ not considered yet (see the discussion in Section 2.1.1). We first note that, from (2.30), a large-rank gauge group (like E_6) can give an acceptable value for the renormalized gauge coupling:

$$\alpha_{GUT}^{-1} \sim C_A \sim \sqrt{N_1}\,. \tag{2.32}$$

At the same time the ratio M_s^2/M_P^2 is normalized down to a value of order $1/N_1$. So, instead of the weak coupling equation (2.11), we have:

$$\frac{M_P^2}{M_s^2} \simeq N_1 \simeq \alpha_{GUT}^{-2}, \tag{2.33}$$

which provide a much better agreement between M_{GUT} and M_s. Explicit string results on the Planck scale renormalization have been carried out by [Kohlprath, 2002]. Generalizing a previous work by [Kiritsis and Kounnas, 1995], he has calculated the one-loop renormalization of the Planck mass in type II string theory compactified on a large class of symmetric orbifolds preserving $\mathcal{N} = 1$ supersymmetry. For certain choices of compactification it has been shown that a rather large one-loop renormalization of the Planck mass is possible.

The other interesting physical consequence that we mention is shared by this model with all induced gravity models. Note in fact that, in the strong coupling limit, gravitational and gauge couplings are determined entirely by loop corrections i.e. are *induced.* Some interest in induced gravity models has been recently re-aroused in contexts such as the study of black hole thermodinamics (see [Wald, 2001] for a recent review) and the discussion of entropy bounds ['t Hooft, 1993] [Veneziano, 1999] [Bousso, 1999]. The two subjects are strongly related to each other (a black hole is in fact supposed to maximize the entropy in a given region of spacetime) and are believed to represent usefull tests for the theories candidate for quantum gravity. Let's start by briefly reviewing how induced gravity ideas may ease some problems related to black hole thermodinamics.

It is known that several classical and semiclassical considerations about black-hole physics conspire toward considering black holes as effective thermodinamical objects with an associated entropy given by

$$S_{\mathrm{bh}} = \frac{A}{4G}, \qquad (D=4) \tag{2.34}$$

A being the area of the event horizon of the black hole. Now, if one tries to calculate the contribution to the black hole entropy of the quantum, non-gravitational fields in their vacuum states in the vicinity of a black hole (either as "entanglement entropy" or "thermal entropy"), one generally finds an infinite result, infinities being provided by space-time integrations near the black hole horizon. Interesting enough, by introducing a UV cut off at the Plank scale, the contribution of each quantum field to the entropy turns out to be of the correct order of magnitude and always proportional to the area of the black hole: $S_{\mathrm{bh}} \sim A/G$. Such a result is, on the other hand, "embarrassing", since *any* quantum field (either known or unknown to physicists!), contributes to the total entropy of the black hole, so that, in order to obtain the correct geometrical result (2.34), one should fine-tune the UV cut off according to the number of independent quantum fields that are being considered. Here is where induced gravity may come into play since, as first noted by [Susskind and Uglum, 1994] and [Jacobson, 1994], in an induced gravity context the effective Newton constant decreases with the number N of species as

$$G_{\mathrm{eff}} \sim \frac{1}{N\,\Lambda^2}. \tag{2.35}$$

This is also evident from (2.30) in the infinite bare coupling limit $\kappa_0 \to \infty$. On the other hand, the appropriate UV cut off is no longer $G_{\mathrm{eff}}^{-1/2}$ but, rather, Λ or again, in string theory, M_s. Thus the correct contribution of each field to the entropy comes to be of order $\Lambda^2 A \simeq A/G_{\mathrm{eff}} N$: the more independent fields are considered the less each contributes! [Frolov, Fursaev and Zelnikov, 1997] have studied in detail the problem of black hole entropy in $D = 4$ dimensions for a specific class of induced gravity models, where several contraints on the number and masses of the matter fields are posed in order to make the result finite (even without a cut off). String-inspired models, however, are already naturally endowed with a UV cut off, the string mass M_s and, in this respect, seem to require much less fine-tuning.

The same reasoning can be easily extended to entropy bound issues. The very appealing idea that (quantum) gravity and quantum field theory may conspire and provide a *geometrical* bound on the total entropy of a given region of spacetime, is threatened by the presence of a large number N of species, each separately contributing to the total entropy. Again, the problem is eased if the Newton constant G itself, converting, so to say, areas into entropy, geometry into information, decreases sufficiently rapidly with the number of different fields N.

Apart from entropy issues, but in the very same spirit, an induced Newton constant can also prevent vacuum gravitational instabilities, another possible effect of the presence of a large number N of fields. As argued by [Brustein, Eichler, Foffa and Oaknin, 2002], in fact, the virtual energy fluctuations of the fields tend to form black holes of sizes increasing with N. For sufficiently large N, the size of the created black hole is large enough in Plank units, so that that region of space-time undergo a (real, not virtual!) gravitational collapse. Once again, a possible recipe is to suppose a relation between G and Λ of the type of equation (2.35).

Chapter 3

Phenomenology of the model

In this chapter we basically review the work done in [Damour, Piazza and Veneziano, 2002] and summarized in [Damour, Piazza and Veneziano, 2002b], dedicated to explore the phenomenological consequences of the the strong coupling scenario described in section 2.3, expecially in terms of equivalence principle violations and space-time variations of couplings.

We briefly summarize the content of Chapter 2 by saying that all string theory models predict the existence of a scalar partner of the spin 2 graviton: the dilaton ϕ, whose vacuum expectation value (VEV) determines the 4-dimensional string coupling constant $g_{4\,s} = e^{\phi/2}$ [Witten, 1984]. At tree level, the dilaton is massless and has gravitational-strength couplings to matter which violate the equivalence principle [Taylor and Veneziano, 1988]. This is in violent conflict with present experimental tests of general relativity [compare, for instance, the estimate (2.21) with the bound (1.70)]. It is generally assumed that this conflict is avoided because, after supersymmetry breaking, the dilaton might acquire a (large enough) mass. However, [Damour and Polyakov, 1994] (see also [Damour and Nordtvedt, 1993]) have proposed a mechanism which can naturally reconcile a *massless* dilaton with existing experimental data. All this is reviewed in Section 2.2. The basic idea of [Damour and Polyakov, 1994] was to exploit the string-loop modifications of the (four dimensional) effective low-energy action

$$S = \int d^4x\sqrt{\widetilde{g}}\left[\frac{B_g(\phi)}{\alpha'}\,\widetilde{R} + \frac{B_\phi(\phi)}{\alpha'}\,[2\widetilde{\Box}\phi - (\widetilde{\nabla}\phi)^2] - \frac{1}{4}\,B_F(\phi)\,\widetilde{F}^2 - V + \cdots\right], \quad (3.1)$$

i.e. the ϕ-dependence of the various coefficients $B_i(\phi)$, $i = g, \phi, F, \ldots$, given in the weak-coupling region ($e^\phi \to 0$) by series of the form

$$B_i(\phi) = e^{-\phi} + c_0^{(i)} + c_1^{(i)}\,e^{\phi} + c_2^{(i)}\,e^{2\phi} + \cdots\,, \quad (3.2)$$

coming from genus expansion of string theory: $B_i = \Sigma_n\, g_s^{2(n-1)} c_n^{(i)}$, with $n = 0, 1, 2, \ldots$. As reviewed in section 2.2.1, [Damour and Polyakov, 1994] have shown that, if there exists a special value ϕ_m of ϕ which extremizes all the (relevant) coupling functions $B_i^{-1}(\phi)$, the cosmological evolution of the graviton-dilaton-matter system naturally drives ϕ towards ϕ_m. This provides a mechanism for fixing a massless dilaton at a value where it decouples from matter ("Least Coupling Principle"). A simple situation where the existence of a universally extremizing dilaton value ϕ_m is guaranteed is that of S duality, i.e. a symmetry $g_s \leftrightarrow 1/g_s$, or $\phi \to -\phi$ (so that $\phi_m = 0$).

The basic assumption of the strong coupling scenario is instead (2.31), suggested by [Veneziano, 2002] after studying the toy model reviewed in section 2.3. In words: the infinite-bare-coupling limit $g_s \to \infty$ ($\phi \to +\infty$) yields smooth *finite* limits for all the coupling functions, namely

$$B_i(\phi) = C_i + \mathcal{O}(e^{-\phi})\,. \tag{3.3}$$

Under this assumption, the coupling functions are all extremized at infinity, i.e. $\phi_m = +\infty$.

In the "large N"-type toy model of [Veneziano, 2002] it would be natural to expect that the $\mathcal{O}(e^{-\phi})$ term in equation (3.3) be *positive*, so that $B_i(\phi)$ be *minimized* at infinity. This would correspond to couplings $\lambda_i(\phi) \sim B_i^{-1}(\phi) = C_i^{-1} - \mathcal{O}(e^{-\phi})$ which are *maximized* at infinity. Note, however, that the most relevant cosmological coupling for this work, the coupling to the inflaton, $\lambda(\phi)$, contained in V (see equation (3.14) below) is closer to a B_i than to its inverse. Thus $\lambda(\phi)$ is naturally *minimized* at infinity (see further discussion of this point below), a crucial property for the attractor mechanism of [Damour and Polyakov, 1994] and [Damour and Nordtvedt, 1993].

This chapter is organized as follows. In Section 3.1 we shall consider in detail the early-time cosmology of models satisfying (3.3). More precisely, our main aims will be to study the efficiency with which a primordial inflationary stage drives ϕ towards the "fixed point" at infinity $\phi_m = +\infty$, thereby generalizing the work [Damour and Vilenkin, 1996] which considered the inflationary attraction towards a local extremum ϕ_m. In Section 3.2 we give quantitative estimates of the present violations of the equivalence principle (non universality of free fall, and variation of "constants") in terms of the parameters introduced in Chapter 1. Our most important conclusion is that the runaway of the dilaton towards strong-coupling (under the assumption (3.3)) naturally leads to WEP violations which are rather large, in the sense of not being much smaller than the presently tested level $\sim 10^{-12}$. This gives additional motivation for the currently planned improved tests of the universality of free fall. Within our scenario, most of the other deviations from general relativity (as the "post-Einsteinian" effects in gravitationally interacting systems described in Section 1.2) are too small to be of phenomenological interest. However, under some assumptions about the coupling of ϕ to dark matter and/or dark energy as those of the cosmological model [Gasperini, Piazza and Veneziano, 2002] of Chapter 4, the time variation of the natural "constants" (notably the fine-structure constant) predicted by our scenario might be large enough to be within reach of improved experimental and/or observational data. The phenomenologically interesting conclusion that equivalence-principle violations are generically predicted to be rather large after inflation (in sharp contrast with the results of [Damour and Vilenkin, 1996]) is due to the fact that the attraction towards an extremum at infinity is much less effective than the attraction towards a (finite) local extremum as originally contemplated by [Damour and Polyakov, 1994]. This reduced effectiveness was already pointed out in by [Damour and Nordtvedt, 1993] within the context of equivalence-principle-respecting tensor-scalar theories (à la Jordan-Fierz-Brans-Dicke).

3.1 Dilaton runaway

In this section we study the dilaton's runaway during the various stages of cosmological evolution. We first show (subsection 3.1.1) that, like in the case of a local extremum [Damour and Vilenkin, 1996], inflation is particularly efficient in pushing ϕ towards the fixed point. We will then argue (subsection 3.1.2) that the order of magnitude of the bare string coupling $e^{\phi} \simeq e^{c\varphi}$ does not suffer further appreciable changes during all the subsequent evolution.

3.1.1 The inflationary period

Assuming some primordial inflationary stage driven by the potential energy of an inflaton field $\widetilde{\chi}$, and taking into account generic couplings to the dilaton ϕ, we consider an effective action of the form

$$S = \int d^4x \sqrt{\widetilde{g}} \left[\frac{B_g(\phi)}{\alpha'} \widetilde{R} + \frac{B_\phi(\phi)}{\alpha'} [2\widetilde{\Box}\phi - (\widetilde{\nabla}\phi)^2] - \frac{1}{2} B_\chi(\phi)(\widetilde{\nabla}\widetilde{\chi})^2 - \widetilde{V}(\widetilde{\chi},\phi) \right] . \quad (3.4)$$

In this string-frame action, the dilaton dependence of all the functions $B_i(\phi)$, $\widetilde{V}(\widetilde{\chi},\phi)$ is assumed to be of the form (3.3). It is convenient to replace the (σ-model) string metric $\widetilde{g}_{\mu\nu}$ by the conformally related Einstein metric introduced in Section 1.1: $g_{\mu\nu} = C\, B_g(\phi)\, \widetilde{g}_{\mu\nu}$, and the dilaton field by the variable

$$\varphi = \int d\phi \left[\frac{3}{4} \left(\frac{B_g'}{B_g} \right)^2 + \frac{B_\phi'}{B_g} + \frac{1}{2} \frac{B_\phi}{B_g} \right]^{\frac{1}{2}} , \; B' \equiv \partial B / \partial \phi \, . \quad (3.5)$$

which differs from (1.3) because of the kinetic term for the dilaton has slightly changed, as explained in Section 2.2. The normalization constant C is chosen so that the string units coincide with the Einstein units when $\phi \to +\infty$: $C\, B_g(+\infty) = 1$. [Note that $C = 1/C_g$ in terms of the general notation of equation (3.3).] Introducing the (modified) Planck mass

$$\widetilde{m}_P^2 = \frac{1}{4\pi G} = \frac{4}{C\alpha'} , \quad (3.6)$$

and replacing also the inflaton by the dimensionless variable $\chi = C^{-1/2}\, \widetilde{m}_P^{-1}\, \widetilde{\chi}$, we end up with an action of the form

$$S = \int d^4x \sqrt{g} \left[\frac{\widetilde{m}_P^2}{4} R - \frac{\widetilde{m}_P^2}{2} (\nabla\varphi)^2 - \frac{\widetilde{m}_P^2}{2} F(\varphi)(\nabla\chi)^2 - \widetilde{m}_P^4\, V(\chi,\varphi) \right] , \quad (3.7)$$

where

$$F(\varphi) = B_\chi(\phi)/B_g(\phi)\,, \;\; V(\chi,\varphi) = C^{-2}\,\widetilde{m}_P^{-4}\,B_g^{-2}(\phi)\,\widetilde{V}(\widetilde{\chi},\phi)\,. \tag{3.8}$$

In view of our basic assumption (3.3), note that, in the strong-coupling limit $\phi \to +\infty$, $d\varphi/d\phi$ tends, according to equation (3.5), to the constant $(C_\phi/2C_g)^{1/2}$, while the dilaton-dependent factor $F(\varphi)$ in front of the inflaton kinetic term tends to the constant C_χ/C_g. The toy model of Ref. [Veneziano, 2002] suggests that the various (positive) constants C_i in equation (3.3) are all largish and comparable to each other. We shall therefore assume that the various ratios C_i/C_j are of order unity. The most important such ratio for the following is $c \equiv (2C_g/C_\phi)^{1/2}$ which gives the asymptotic behaviour of the bare string coupling as

$$g_s^2 = e^\phi \simeq e^{c\varphi}\,. \tag{3.9}$$

In view of the fact that, in the strong-coupling limit we are interested in, the factor $F(\varphi)$ in equation (3.7) quickly tends to a constant, we can simplify our analysis (without modifying the essential physics) by replacing it by a constant (which can then be absorbed in a redefinition of χ). Henceforth, we shall simply take $F(\varphi) = 1$. [See, however, the comments below concerning the self-regenerating inflationary regime.]

Following [Damour and Nordtvedt, 1993] it is then useful to combine the Friedmann equations for the scale factor $a(t)$ during inflation ($ds^2 = -dt^2 + a^2(t)\,\delta_{ij}\,dx^i\,dx^j$) with the equations of motion of the two scalar fields $\chi(t)$, $\varphi(t)$, to write an autonomous equation describing the evolution of the two scalars in terms of the parameter

$$p = \int H dt = \int \frac{\dot{a}}{a}\,dt = \ln\,a + \text{const} \tag{3.10}$$

measuring the number of e-folds of the expansion. For any multiplet of scalar fields, $\boldsymbol{\varphi} = (\varphi^a)$, this yields the simple equation [Damour and Nordtvedt, 1993]

$$\frac{2}{3-\boldsymbol{\varphi}'^2}\,\boldsymbol{\varphi}'' + 2\,\boldsymbol{\varphi}' = -\nabla_{\boldsymbol{\varphi}} \ln|V(\boldsymbol{\varphi})|\,, \tag{3.11}$$

where $\boldsymbol{\varphi}' \equiv d\boldsymbol{\varphi}/dp$, and where all operations on $\boldsymbol{\varphi}$ are covariantly defined in terms of the σ-model metric $d\sigma^2 = \gamma_{ab}(\varphi)\,d\varphi^a\,d\varphi^b$ defining the scalar kinetic terms. In our simple model (with $F(\varphi) = 1$), we have a flat metric $d\sigma^2 = d\varphi^2 + d\chi^2$. [Note that, when $\gamma_{ab}(\varphi)$ is curved the acceleration term $\boldsymbol{\varphi}''$ involves a covariant derivative.]

As noted by [Damour and Nordtvedt, 1993], the generic solution of equation (3.11) is easily grasped if one interprets it as a mechanical model: a particle with position $\boldsymbol{\varphi}$, and velocity-dependent mass $m(\boldsymbol{\varphi}') = 2/(3-\boldsymbol{\varphi}'^2)$, moves, in the "time" $p = \ln a + \text{const}$, in the manifold $d\sigma^2$ under the influence of an external potential $\ln|V(\boldsymbol{\varphi})|$ and a constant friction force $-2\,\boldsymbol{\varphi}'$. If the curvature of the effective potential $\ln|V(\boldsymbol{\varphi})|$ is sufficiently small the motion of $\boldsymbol{\varphi}$ rapidly becomes slow and friction-dominated:

$$2\,\frac{d\boldsymbol{\varphi}}{dp} \simeq -\nabla_{\boldsymbol{\varphi}} \ln V(\boldsymbol{\varphi})\,. \tag{3.12}$$

equation (3.12) is equivalent to the usual "slow roll" approximation.

Consistently with our general assumption (3.3), we consider potentials allowing a strong-coupling expansion of the form:

$$V(\chi,\varphi) = V_0(\chi) + V_1(\chi)e^{-c\varphi} + \mathcal{O}(e^{-2c\varphi})\,, \tag{3.13}$$

where $V_0(\chi)$ is a typical chaotic-inflation potential with $V_0(0) = 0$, while $V_1(0) = v_1 \geq 0$ can possibly provide (if $v_1 > 0$) the effective cosmological constant driving today's acceleration in the scenario of [Gasperini, Piazza and Veneziano, 2002], which is the subjetc of next chapter. For the sake of simplicity we shall discuss mainly the "factorized" power-law case $V_0(\chi) \sim V_1(\chi) \sim \chi^n$ for which we can conveniently write V in the form:

$$V(\chi,\varphi) = \lambda(\varphi)\,\frac{\chi^n}{n}\,, \tag{3.14}$$

with a dilaton-dependent coupling constant $\lambda(\varphi)$ of the form

$$\lambda(\varphi) = \lambda_\infty(1 + b_\lambda\, e^{-c\varphi})\,. \tag{3.15}$$

This example belongs to the class of the two-field inflationary potentials discussed in [Linde, 1990]. We have checked that our results remain qualitatively the same for the more general potential (3.13) provided that $V_0(\chi)$ and $V_1(\chi)$ are not extremely diffent and given the fact that v_1 is phenomenologically constrained to be very small. Note that, within the simplified model (3.15), the ratio $V_1(\chi)/V_0(\chi)$ is equal to the constant coefficient b_λ.

The universal (positive) constant c appearing in the exponential $e^{-c\varphi}$ is the same as in equation (3.9) [i.e. $c \equiv (2C_g/C_\phi)^{1/2}$, which is expected to be of order unity]. The coefficient b_λ in (3.15) is such that $b_\lambda\, e^{-c\varphi} \simeq b_\lambda\, e^{-\phi}$ roughly corresponds to a combination of terms $\sim \pm\, C_i^{-1}\, \mathcal{O}(e^{-\phi})$ coming from the strong-coupling asymptotics of several $B_i(\phi)$, equation (3.3) (see equation (3.8)). In the toy model of section 2.3 [Veneziano, 2002] one would therefore expect b_λ to be smallish. Anyway, we shall see that in final results only the ratios of such b_i coefficients enter. More important than the magnitude of b_λ is its sign. It is crucial for the present strong-coupling attractor scenario to assume that $b_\lambda > 0$, i.e that $\lambda(\varphi)$ reaches a *minimum* at strong-coupling, $\varphi \to +\infty$. Note again that this behaviour is consistent with the simple "large N"-type idea of [Veneziano, 2002] if we assimilate $\lambda(\varphi)$ to one of the inverse couplings B_i appearing in (3.1) (for instance $B_F \sim g_F^{-2}$, where g_F is a gauge coupling), rather than to the coupling itself. If the latter were the case, $\lambda(\varphi)$ would reach a *maximum* as $\phi \to +\infty$, and the attractor mechanism of [Damour and Polyakov, 1994] would drive ϕ towards weak coupling ($\phi \to -\infty$). However, the Einstein-frame ϕ-dependence of $V(\chi)$ gets contributions from several $B_i^{\pm n}(\phi)$, equation (3.8), which might conspire to minimize it at strong coupling. This feature is also probably necessary in order to solve the cosmological-constant problem through some argument by which the vacuum at infinity has vanishing energy density.

Substituting the potential (3.13) into the slow roll equation (3.12) and assuming (for simplicity) that $V_1(\chi)e^{-c\varphi}$ is significantly smaller than $V_0(\chi)$ leads to a decoupled set of evolution equations for χ and φ (where $V' \equiv \partial V/\partial\chi$):

$$\frac{d\chi}{dp} = -\frac{1}{2}\frac{V_0'}{V_0}\,, \tag{3.16}$$

$$\frac{d\varphi}{dp} = \frac{1}{2}\,c\,e^{-c\varphi}\frac{V_1}{V_0}\,. \tag{3.17}$$

Given some "initial" conditions $\chi_{\rm in}$, $\varphi_{\rm in}$ (discussed below) at some starting point, say $p = 0$, the solution of equations (3.16), (3.17) is simply

$$p = 2\int_{\chi}^{\chi_{\rm in}} d\bar\chi\,\bar\chi\left(\frac{V_0(\bar\chi)}{\bar\chi\,V_0'(\bar\chi)}\right)\,, \tag{3.18}$$

$$e^{c\varphi} = e^{c\varphi_{\rm in}} + \frac{c^2}{2}\int dp\,\frac{V_1(\chi(p))}{V_0(\chi(p))}\,, \tag{3.19}$$

which simply become:

$$p = \frac{1}{n}\left(\chi_{\rm in}^2 - \chi^2\right)\,,$$
$$e^{c\varphi} + \frac{b_\lambda\,c^2}{2n}\,\chi^2 = \text{const.} = e^{c\varphi_{\rm in}} + \frac{b_\lambda\,c^2}{2n}\,\chi_{\rm in}^2\,, \tag{3.20}$$

in the simplified case of eqs. (3.14), (3.15).

Equations (3.20) show that, in order for the string coupling $g_s^2 \simeq e^{c\varphi}$ to have reached large values at the end of inflation, a large total number of e-folds must have occurred while the (dimensionless) inflaton field χ decreases from a large initial value, to a value of order unity (in Planck units). To get a quantitative estimate of the string coupling at the end of inflation we need to choose the initial conditions $\chi_{\rm in}$, $\varphi_{\rm in}$. A physically reasonable way (which is further discussed below) of choosing $\chi_{\rm in}$ is to start the classical evolution (3.16)-(3.20) at the exit of the era of self-regenerating inflation (see [Linde, 1990] and references therein). We will now show how to relate the exit from self-regenerating inflation to the size of density fluctuations generated by inflation.

Let us recall (see [Linde, 1990] and references therein) that the density fluctuation $\delta \equiv \delta\rho/\rho$ on large scales (estimated in the one-field approximation where the inflaton χ is the main contributor) is obtained by evaluating the expression

$$\delta(\chi) \simeq \frac{4}{3}\frac{1}{\pi}\left(\frac{2}{3}\right)^{\frac{1}{2}}\frac{V^{3/2}}{\partial_\chi V} \tag{3.21}$$

at the value $\chi = \chi_\times$, at which the physical scale we are considering crossed the horizon outwards during inflation. For the scale corresponding to our present horizon this usually corresponds to a value $\chi_\times(H_0)$ (χ_H for short) reached some 60 e-folds before the end of slow-roll. Following [Linde, 1990] we have $\chi_H \simeq 5\sqrt{n}$ for the model (3.14) (and with our modified definition of χ). The numerical value of $\delta_H \equiv \delta(\chi_H)$ which is compatible with cosmological data (structure formation and cosmic microwave background) is $\delta_H \simeq 5 \times 10^{-5}$. In the model (3.14) the function $\delta(\chi)$ defined by (3.21) scales with χ as $\chi^{\frac{n+2}{2}}$. Putting together this information we obtain a relation between χ_{in} and $\delta(\chi_{\text{in}})$, which involves the value of the observable horizon-size fluctuations $\delta_H \equiv \delta(\chi_H)$:

$$\frac{\delta(\chi_{\text{in}})}{\delta(\chi_H)} = \left(\frac{\chi_{\text{in}}}{\chi_H}\right)^{\frac{n+2}{2}}, \tag{3.22}$$

i.e.

$$\chi_{\text{in}} \simeq \chi_H \left(\frac{\delta_{\text{in}}}{\delta_H}\right)^{\frac{2}{n+2}} \simeq 5\sqrt{n}\left(\frac{\delta_{\text{in}}}{\delta_H}\right)^{\frac{2}{n+2}}, \tag{3.23}$$

where we introduced the short-hand notation $\delta_{\text{in}} \equiv \delta(\chi_{\text{in}})$.

Inserting equation (3.23) into equation (3.19) we then obtain the following estimate of the string coupling constant after inflation as a function of φ_{in} and $\delta(\chi_{\text{in}})$

$$\begin{aligned} e^{c\varphi_{\text{end}}} - e^{c\varphi_{\text{in}}} &\simeq \frac{c^2}{2}\langle V_1/V_0\rangle p \\ &\sim \frac{c^2}{2n}\langle V_1/V_0\rangle \chi_{\text{in}}^2 \sim \frac{25c^2}{2}\langle V_1/V_0\rangle \left(\frac{\delta_{\text{in}}}{\delta_H}\right)^{\frac{4}{n+2}}, \end{aligned} \tag{3.24}$$

where $\langle V_1/V_0\rangle$ denotes the average value of V_1/V_0: $\langle V_1/V_0\rangle \equiv \int dp(V_1/V_0)/\int dp$ [note that this average ratio is equal to b_λ in the simplified model (3.15)].

To get a quantitative estimate of $e^{c\varphi_{\text{end}}}$ we still need to estimate the value of $\delta(\chi_{\text{in}})$ corresponding to the chosen "initial" value of the inflaton. As we will now check, taking for χ_{in} the value corresponding to the exit from self-regenerating inflation corresponds simply to taking $\delta(\chi_{\text{in}}) \sim 1$. Indeed, let us first recall that, during inflation, each (canonically normalized) scalar field (of mass smaller than the expansion rate H) undergoes typical quantum fluctuations of order $H/(2\pi)$, per Hubble time [Linde, 1990]. This implies (for our dimensionless fields) that the value of χ at the exit from self-regeneration, say χ_{ex}, is characterized by $\widehat{H}_{\text{ex}}/(2\pi) \approx [\partial_\chi V/(2V)]_{\text{ex}}$, where $\widehat{H} \equiv H/\widetilde{m}_P$ is the dimensionless Hubble expansion rate and where the right-hand side (RHS) is the classical change of χ per Hubble time (corresponding to the RHS of equation(3.16)). Using Friedmann's equation (in the slow-roll approximation) $\widehat{H}_{\text{ex}}^2 \approx (2/3)V(\chi_{\text{ex}})$, it is easily seen that that the exit from self-regeneration corresponds to $\delta(\chi_{\text{ex}}) \approx 4/3 \sim 1$. It is, a posteriori, physically quite reasonable to start using the classical evolution system only when the (formal extrapolation) of the density fluctuation $\delta(\chi)$ becomes smaller than one.

Within some approximation, one can implement the effect of the combined quantum fluctuations of (φ, χ) by adding random terms with r.m.s. values $\widehat{H}/2\pi$ on the right hand side of equations (3.16) and (3.17), $d\chi/dp$ and $d\varphi/dp$ being precisely the shifts of the fields in a Hubble time. The system of equations becomes thus of the Langevin-type

$$\frac{d\chi}{dp} = -\frac{1}{2}\frac{V_0'}{V_0} + \frac{\widehat{H}}{2\pi}\xi_1, \tag{3.25}$$

$$\frac{d\varphi}{dp} = \frac{1}{2} c\, e^{-c\varphi} \frac{V_1}{V_0} + \frac{\widehat{H}}{2\pi} \xi_2 , \tag{3.26}$$

where $\widehat{H} \approx [(2/3)V(\chi,\phi)]^{1/2}$ (in the slow-roll approximation) is the dimensionless expansion rate, and where ξ_1 and ξ_2 are (independent) normalized random white noises:

$$\langle \xi_i(p_1)\, \xi_j(p_2) \rangle = \delta_{ij}\delta(p_1 - p_2), \qquad i,j = 1,2 . \tag{3.27}$$

When the random force terms dominate the evolution in either equation (3.25) or equation (3.26) the quasi-classical description (3.16), (3.17) breaks down. The phase space of the sytem can thus be roughly divided into four regions according to whether the evolution of none, one or both of the two fields is dominated by quantum fluctuations. This is depicted in Figure 1 where such regions are delimited by dashed, thick curves in the case of a power-law potential (3.14).

Apart from factors of order one, the evolution of the inflaton χ is quasi-classical in the region under the line $\chi = \lambda_\infty^{-1/(n+2)}$. In the chaotic inflationary models [Linde, 1990] such an inflaton's value corresponds to the exit from the self-regenerating regime and to the beginning of the quasi-classical slow-roll inflation. As mentioned above it also corresponds to a perturbation $\delta(\chi) \sim 1$. Somewhat surprisingly, in the model at hand, however, the quasi-classical region for the inflaton evolution $\chi \lesssim \lambda_\infty^{-1/(n+2)}$ is affected by quantum fluctuations that still dominate the evolution of φ in the region above the hyperbola-like curve $\chi = b_\lambda^{2/n} \lambda_\infty^{-1/n} e^{-2c\varphi/n}$. We must therefore study, in some detail, the evolution of the system in the presence of the noise term for φ as in eq. (3.26), assuming a quasi classical evolution for χ. This is done in Appendix C. The final result is that, if we start at $\chi \lesssim \lambda_\infty^{-1/(n+2)}$, (classical evolution for χ) the average value of $e^{c\varphi}$ is *multiplicatively renormalized*, by a factor of order unity, with respect to the classical trajectory $e^{c\varphi_{\rm cl}}$, given by solving equations (3.16) - (3.17), i.e. $\langle e^{c\varphi} \rangle = \mathcal{O}(1) e^{c\varphi_{\rm cl}}$. One also finds that the dispersion of $e^{c\varphi}$ around its average value is comparable to its average value.

We shall not try to discuss here what happens in the self-regenerating region $\chi \gtrsim \chi_{\rm in} \sim \lambda_\infty^{-1/(n+2)}$. Let us recall that the simple decoupled system (3.16)-(3.17) was obtained by neglecting the kinetic coupling term $F(\varphi)$ in equation (3.7). If we were to consider a more general model, we would have more coupling between χ and φ and we would expect that (contrary to Fig. 1 which exhibits a "classical φ region" above the "quantum χ line") the evolution in the self-regenerating region involves a strongly coupled system of Langevin equations. Then, as discussed in [Linde, 1990], solving such a system necessitates to give boundary conditions on all the boundaries of the problem: notably for $\chi \to \infty$, but also for $\varphi \to +\infty$ and $\varphi \to -\infty$. We leave to future work such an investigation (and a discussion of what are reasonable boundary conditions). In this work we shall content ourselves with "starting" the evolution on the quantum χ boundary line $\chi_{\rm in}$ with some value $\varphi = \varphi_{\rm in}$, assuming that $e^{c\varphi_{\rm in}}$ is smaller than the driving effect due to inflation, i.e. than the RHS of equation (3.24). [This assumption is most natural in a work aimed at studying the "attracting" effect due to primordial inflation.]

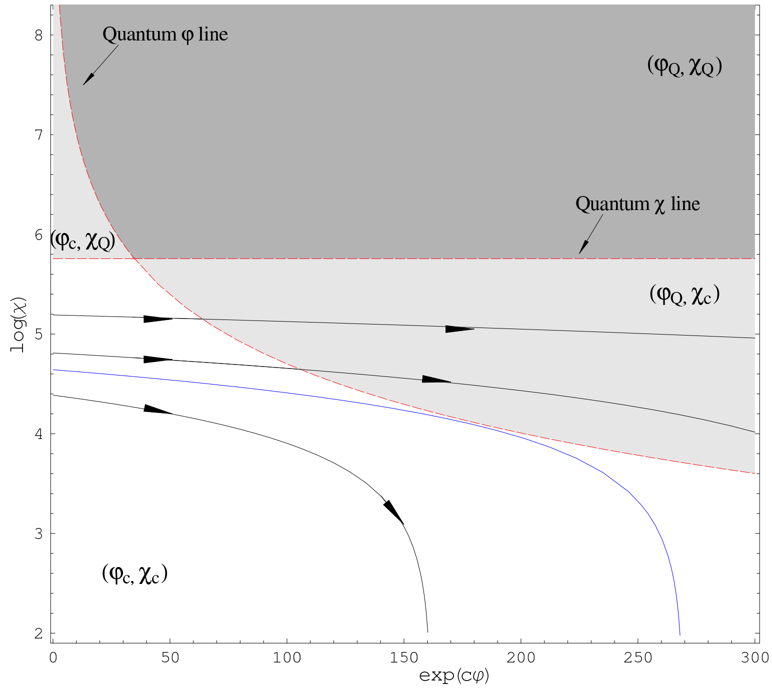

Figure 3.1: *The phase space of the system is represented in the case of a power–law potential (3.14) with $n = 2$, $b_\lambda = 0.1$ and $\lambda_\infty = 10^{-10}$. The thick–dashed (red) curves delimitate the quantum behaviour of the two fields, the horizontal curve $\chi = \lambda_\infty^{-1/(n+2)}$ and the hyperbola-like curve $\chi = b_\lambda^{2/n} \lambda_\infty^{-1/n} e^{-2c\varphi/n}$ being the limit of the quantum behaviour for χ and φ respectively. In the white region both fields have a classical behaviour. The last "fully classical" trajectory has been represented by a thick (blue) curve. The bright–gray regions are those where either the φ or the χ evolution are dominated by quantum fluctuations. The fully–quantum region is the dark–gray region on the top right.*

Going back to our result (3.24), we can now insert, according to the preceding discussion, the values $\delta_{\rm in} = 1$ and $e^{c\varphi_{\rm in}} \ll e^{c\varphi_{\rm end}}$. Finally, in the simplified model (3.14), (3.15), we get the estimate:

$$e^{c\varphi_{\rm end}} = \mathcal{O}(1) \cdot e^{c\varphi_{\rm cl,\,end}} \sim \mathcal{O}(1) \cdot \frac{25c^2}{2} b_\lambda \, (\delta_H)^{-\frac{4}{n+2}} . \qquad (3.28)$$

A more general analysis, based on the potential (3.13) leads to the same final result but with n replaced by some average value of $\chi V_{0,\chi}/V_0$, and with b_λ replaced by some average of the ratio V_1/V_0. Note that smaller values of the exponent n lead to larger values of $e^{c\varphi_{\rm end}}$, i.e. to a more effective attraction towards the "fixed point at infinity". The same is true if we take different exponents n_0 and n_1 (for V_0 and V_1 respectively) and assume $V_1(\chi_{\rm in}) \gg V_0(\chi_{\rm in})$ to hold as a result of $\chi_{\rm in} \gg 1$ and $n_1 > n_0$. Also note that, numerically, if we consider $n = 2$, i.e. the simplest chaotic-inflation potential $V = \frac{1}{2} m_\chi^2(\varphi)\, \chi^2$, equation (3.28) involves the large number $12.5 \times \delta_H^{-1} \sim 2.5 \times 10^5$. In the case where $n = 4$, i.e. $V = \frac{1}{4}\lambda(\varphi)\chi^4$, we have instead the number $12.5 \times \delta_H^{-2/3} \sim 0.92 \times 10^4$. To understand the phenomenological meaning of these numbers we need to relate $e^{c\varphi_{\rm end}}$ to the present, observable deviations from general relativity. This issue is addressed in Section 3.2 after having argued that the post-inflationary evolution of φ is sub-dominant.

3.1.2 Attraction of φ by the subsequent cosmological evolution

We have discussed above the efficiency with which inflation drives the dilaton towards a fixed point at infinity. We need to complete this discussion by estimating the effect of the many e-folds of expansion that took place between the end of inflation and the present time. To address this question, we need to study in more detail the coupling of a runaway dilaton to various types of matter, say a multi-component distribution of (relativistic or non-relativistic) particles. This has already been done in Section 1.3 in the field formalism. Equivalently, one can write the classical action for particles. Working in the Einstein frame,

$$S = \int d^4x\sqrt{g}\left[\frac{\widetilde{m}_P^2}{4}R - \frac{\widetilde{m}_P^2}{2}(\nabla\varphi)^2 - \frac{1}{4}B_F(\varphi)F^2 + \dots\right] - \sum_A \int m_A[\varphi(x_A)]\sqrt{-g_{\mu\nu}(x_A)dx_A^\mu dx_A^\nu}\,. \tag{3.29}$$

As in (1.94) we introduce the crucial dimensionless quantity

$$\alpha_A(\varphi) \equiv \frac{\partial \ln m_A(\varphi)}{\partial\varphi}\,, \tag{3.30}$$

measuring the coupling of φ to a particle of type A. [For consistency with previous work, we keep the notation α_A but warn the reader that this should not be confused with the various gauge coupling constants, often denoted $\alpha_i = g_i^2/4\pi$.] The quantity α_A determines the effect of cosmological matter on the evolution of φ through the general equation [Damour and Nordtvedt, 1993, Damour and Polyakov, 1994]

$$\frac{2}{3-\varphi'^2}\,\varphi'' + \left(1 - \frac{P}{\rho}\right)\varphi' = -\sum_A \alpha_A(\varphi)\,\frac{\rho_A - 3P_A}{\rho}\,, \tag{3.31}$$

where the primes denote derivatives with respect to $p = \ln a + \text{const}$ and where $\rho = \Sigma_A\,\rho_A$ and $P = \Sigma_A\,P_A$ are the total "material" energy density and pressure respectively, both obtained as sums over the various components filling the universe *at the exception of* the kinetic energy density and pressure of φ, $\rho_k = (\widetilde{m}_P^2/2)(d\varphi/dt)^2 = (\widetilde{m}_P^2/2)H^2\varphi'^2$ and $P_k = \rho_k$. Accordingly, the Friedmann equation reads

$$3H^2 = \frac{2}{\widetilde{m}_P^2}\,\rho_{\rm tot} = \frac{2\,\rho}{\widetilde{m}_P^2} + H^2\varphi'^2. \tag{3.32}$$

Note that ρ and P may also account for the potential energy density and pressure of the scalar field, $\rho_V = V(\varphi)$, $P_V = -\rho_V$ and that one can formally extend equation (3.31) to the "vacuum energy" component $V(\varphi)$ by associating to the potential $V(\varphi)$ the mass scale $m_V(\varphi) \equiv V(\varphi)^{\frac{1}{4}}$ which gives $\alpha_V = \partial \ln m_V(\varphi)/\partial\varphi = \frac{1}{4}\partial \ln V(\varphi)/\partial\,\varphi$. Equation (3.11) is then recovered in the limit where the scalar field is the dominant component.

In the simple cases (which are quite frequent; at least as approximate cases) where one "matter" component, with known "equation of state" $P_A/\rho_A = w_A =$ const., dominates the cosmological density and pressure, equation (3.31) yields an autonomous equation for the evolution (with redshift) of φ. Using (3.32) one finds that the "equation of state parameter" $w_{\rm tot} \equiv P_{\rm tot}/\rho_{\rm tot}$ corresponding to the *total* energy and pressure (including now the kinetic contributions of φ; i.e. $\rho_{\rm tot} = \rho + (\widetilde{m}_P^2/2)(d\varphi/dt)^2$, $P_{\rm tot} = P + (\widetilde{m}_P^2/2)\,(d\varphi/dt)^2$) is given in terms of the "matter" equation-of-state parameter $w \equiv P/\rho$ by

$$w_{\rm tot} = w + \frac{1-w}{3}(\varphi')^2 \,. \tag{3.33}$$

The knowledge of $w_{\rm tot}$ then allows one to write explicitly the energy-balance equation $d\rho_{\rm tot} + 3(\rho_{\rm tot} + P_{\rm tot})d\ln a = 0$, which is easily solved in the simple cases where $w_{\rm tot}$ is (approximately) constant.

We see from equation (3.31) that, during the radiation era (starting, say, immediately after the end of inflation), i.e. when the universe is dominated by an ultra-relativistic gas ($\rho_A - 3P_A = 0$), the "driving force" on the right-hand side of (3.31) vanishes, so that φ is not driven further away towards infinity. Actually, one should take into account both the "inertial" effect of the "velocity" φ' acquired during the preceding inflationary driving of φ, and the integrated effect of the many "mass thresholds", $T_A \sim m_A$, when some component becomes non-relativistic (so that $\rho_A - 3P_A \neq 0$). Using the results of [Damour and Nordtvedt, 1993, Damour and Polyakov, 1994] one sees that, in our case, both these effects have only a small impact on the value of φ. Therefore, to a good approximation $\varphi \simeq \varphi_{\rm end}$ until the end of radiation era.

On the other hand, when the universe gets dominated by non-relativistic matter, one gets a non-zero driving force in equation (3.31). In the slow roll approximation, as the transient behaviour has died out, since $w = P/\rho$ gets negligible, we have simply

$$\varphi'_m = -\alpha_m(\varphi)\,, \tag{3.34}$$

where φ'_m stands the φ-velocity during matter domination. and $\alpha_m(\varphi)$ denotes the coupling (3.30) to dark matter.

The coupling to dark matter, $\alpha_m(\varphi)$, depends on the assumption one makes about the asymptotic behaviour, at strong bare string coupling, of the mass of the WIMPs constituting the dark matter. One natural looking, minimal assumption is that dark matter, like all visible types of matter, is coupled in a way which levels off at strong-bare-coupling, as in equation (3.3). In other words, one generally expects that $m_m(\varphi) \simeq m_m(+\infty)(1 + b_m\, e^{-c\varphi})$ so that $\alpha_m(\varphi) \simeq -b_m\, c\, e^{-c\varphi}$. It is then easy to solve equation (3.34), with initial conditions $\varphi_0 = \varphi_{\rm end}$, $\varphi'_0 = 0$ (inherited from radiation era) at the beginning of the matter era. But we shall not bother to write the explicit solution because it is easily seen that the smallness of $e^{-c\varphi_{\rm end}}$ guarantees that the "driving force" $\propto \alpha_m(\varphi)$ remains always so small that the $\mathcal{O}(10)$ e-folds of matter era until vacuum-energy domination (or until the present) have only a fractionally negligible effect on φ.

A more significant evolution of φ during the matter era is provided if, as first proposed in [Damour, Gibbons and Gundlach, 1990] and taken up in [Sandvik, Barrow and Magueijo, 2002, Olive and Pospelov, 2002], dark matter couples much more strongly to φ than "ordinary" matter. Such a stronger coupling to dark matter, which is not constrained by usual equivalence principle experiments, follows assuming more general quantum corrections in the dark matter sector of the theory, i.e. corrections such that the dark matter mass $m_m(\varphi)$, instead of levelling off, either vanishes or keeps increasing at strong bare coupling: $m_m(\varphi) \propto e^{c_m\varphi}$, so that $\alpha_m = c_m$ is a (negative or positive) constant. In [Gasperini, Piazza and Veneziano, 2002] (but see also [Amendola and Tocchini-Valentini, 2001]) it has been shown that under the latter assumption (i.e. with a positive coupling parameter $\alpha_m > 0$) the dilaton can play the role of quintessence, leading to a late-time cosmology of accelerated expansion. By equation (3.34) we have $\varphi = \varphi_{\rm end} - \alpha_m p$, where p is now counted from the end of the radiation era. Given that about nine e-folds separate us from the end of the radiation era, we see that such an evolution might (if $|\alpha_m|$ is really of order unity) have a significant effect on the present value of φ (when compared with the value at the end of inflation, i.e. $c\varphi_{\rm end} \sim \ln(1/\delta_H) \sim 10$). However, the running of φ during the matter era changes the standard recent cosmological picture and is therefore constrained by observations. In fact, by equation (3.33), the total matter-era equation of state parameter $w_{\rm tot}$ in the presence of the dilaton reads $w_{\rm tot} = (\varphi'_m)^2/3$. Accordingly, the matter density varies as $\rho \propto a^{-3(1+w_{\rm tot})} = a^{-(3+(\varphi'_m)^2)}$, possibly affecting the standard scenario of structure formation as well as the global temporal picture between now and the epoch of matter-radiation equality. The compatibility with phenomenology therefore puts constraints on the magnitude of ${\varphi'_m}^2 = \alpha_m(\varphi)^2$. In [Gasperini, Piazza and Veneziano, 2002] $w_{\rm tot} < 0.1$, i.e. $\alpha_m^2 < 0.3, \alpha_m < 0.54$, was suggested to be the maximal deviation one can roughly tolerate during the matter era. More recently, the constraints on the structure formation in dilatonic quintessence model have been investigated by [Amendola *et al.*, 2002]. For a last accelerating period starting out at redshift $z \simeq 1$ the bound is in fact more stringent: $\alpha_m < 0.3$. Curiously enough, [Amendola *et al.*, 2002] find also that in the quintessential model of [Gasperini, Piazza and Veneziano, 2002] to be considered in the next chapter, the accelerating phase may start as early as $z \simeq 5$. If it starts at redshifts larger that $z \simeq 1.5$ then a lower bound on α'_m is also needed! The intuitive reason is that the contribution of the dilaton to the gravitational force of non relativistic matter ease the formation of structure.

In any case, the displacement of φ during matter era smaller than the dispersion $\varphi_{\rm end} - \varphi_{\rm cl,end} \sim \varphi_{\rm cl,end}$ produced by quantum fluctuations during inflation, equation (3.28).

In this context one should also consider the attraction effect of a negative pressure component, either in the form of a φ-dependent vacuum energy (dilatonic quintessence) or in the form of any other, φ-independent component (such as a "genuine" cosmological constant). Of course, the present recent ($z \lesssim 1$) accelerated expansion phase is very short (in "p-time") and sensible changes of the dilaton value since the end of matter domination are not expected. Still, it is crucial to estimate the present dilaton velocity φ_0' since it is related to the cosmological variations of the coupling constants (see next section). In the general case where both non-relativistic matter and (possibly φ-dependent) vacuum energy density $V(\varphi)$ are present, the value of φ_0' predicted by our model is obtained by applying equation (3.31) (in the slow-roll approximation):

$$(\Omega_m + \Omega_V)(1 - w_0)\varphi_0' = (\Omega_m + 2\Omega_V)\varphi_0' = -\Omega_m \alpha_m - 4\Omega_V \alpha_V. \tag{3.35}$$

In the above expression Ω_m and Ω_V are the non-relativistic (dark) matter- and the vacuum-fraction of critical energy density $\rho_c \equiv (3/2)\widetilde{m}_P^2 H^2$ respectively, and the already mentioned prescriptions $\alpha_V = \frac{1}{4}\partial \ln V(\varphi)/\partial\varphi$, $P_V = -\rho_V = -V(\varphi)$ have been used.

The value of φ_0' is therefore some combination of the values of α_m and α_V. We can have two classes of contrasting situations: In the first class, the dilaton couples "normally" (i.e. weakly) both to dark matter and to dark energy, i.e. both $\alpha_m \simeq -b_m c\, e^{-c\varphi} \ll 1$ and $\alpha_V \ll 1$ and equation (3.35) implies $\varphi_0' \ll 1$. In the second class, the dilaton couples more strongly to some type of dark matter or energy, i.e. either (or both) α_m or/and α_V is of order unity so that $\varphi_0' = \mathcal{O}(1)$. The second case is realized in the scenario of [Gasperini, Piazza and Veneziano, 2002]. In the context of this scenario we have an exponential dependence of the potential on φ, $V(\varphi) \simeq V_1 e^{-c\varphi}$ so that $\alpha_V \simeq -(c/4)$ and

$$\varphi_0' = \frac{c\,\Omega_V - \alpha_m \Omega_m}{2\Omega_V + \Omega_m} \lesssim \frac{c}{3}. \tag{3.36}$$

The last inequality follows from the bound $\alpha_m > c/2$ (which is a necessary condition to have positive acceleration in the model of [Gasperini, Piazza and Veneziano, 2002]) and the reasonable bound $\Omega_m > 0.25$.

In the present work, we wish, however, to be as independent as possible from specific assumptions (as the ones used by [Gasperini, Piazza and Veneziano, 2002]). Therefore, rather than insisting on specific (model-dependent) predictions for the present value of φ_0' we wish to find the (model-independent) upper bounds on the possible values of φ_0' set by current observational data. There are several ways of getting such phenomenological bounds, because the existence of a kinetic energy (and pressure) associated to $d\varphi/dt = H\varphi'$ has several observable consequences. A rather secure bound can be obtained by relating the value of φ' to the deceleration parameter $q \equiv -\ddot{a}a/\dot{a}^2$. In the general class of models that we consider, the cosmological energy density and pressure have (currently) three significant contributions: dark matter ($\Omega_m = \rho_m/\rho_c$), dark energy (Ω_V) and the kinetic effect of a scalar field ($\Omega_k = \rho_k/\rho_c$ with $\rho_k = (\widetilde{m}_P^2/2)(d\varphi/dt)^2 = (\widetilde{m}_P^2/2)H^2\varphi'^2$ so that $\Omega_k = \varphi'^2/3$). We assume (consistently with recent cosmic background data) that the space curvature is zero. Therefore we have the first relation

$$\Omega_m + \Omega_V + \Omega_k = 1 = \Omega_m + \Omega_V + \varphi'^2/3\,. \tag{3.37}$$

The deceleration parameter is given by the general expression $2q = \Sigma_A \Omega_A (1 + 3w_A)$. Using $w_m = 0$, $w_V = -1$ and $w_k = +1$, we get

$$2q = \Omega_m - 2\Omega_V + \frac{4}{3}\varphi'^2. \qquad (3.38)$$

Using the relation (3.37) above to eliminate Ω_V we get the following expression for φ'^2 in terms of the observable quantities q and Ω_m

$$\varphi'^2 = 1 + q - \frac{3}{2}\Omega_m. \qquad (3.39)$$

The supernovae Ia data [A. Riess *et al*, 1998; Perlmutter *et al.*, 1999] give a strict upper bound on the present value q_0: $q_0 < 0$. A generous lower bound on the present value of Ω_m is $\Omega_{m0} > 0.2$ [1]. Inserting these two constraints in equation(3.39) finally yields the safe upper bound

$$\varphi_0'^2 < 0.7\,, \text{ i.e. } |\varphi_0'| < 0.84\,. \qquad (3.40)$$

To summarize, quite different rates of evolution for the dilaton are possible. A very slow variation is expected whenever dilaton couplings to both dark energy and dark matter follow the "normal" behaviour (3.3). Otherwise, dilaton variations on the Hubble scale are expected. However, cosmological observations set the strict upper bound (3.40) on the present time variation of φ. For the purpose of the present section (evaluating the current location of the dilaton) these two alternatives do not make much difference because the vacuum-dominance era has started less than about 0.7 e-folds away ($\ln(1 + z_*)$ with $z_* < 1$). Therefore, φ did not have enough "p-time" , during vacuum dominance, to move much, even if it is coupled to vacuum energy with $\alpha_V \simeq -(c/4) \sim 1$.

Finally, we conclude from this analysis that, to a good approximation (and using the fact that the phenomenology of the matter-era constrains the dark-matter couplings of the dilaton to be rather small), the value of φ now is essentially given by the value φ_{end} at the end of inflation, i.e. by equation (3.24).

3.2 Deviations from general relativity induced by a runaway dilaton

3.2.1 Composition-independent deviations from general relativity

The previous section has reached the conclusion that present deviations from general relativity are given, to a good approximation, by the values of the matter-coupling coefficients $\alpha_A(\varphi)$ given by equation (3.30) calculated at $\varphi \simeq \varphi_{\text{end}}$ as given by equation (3.28). Let us now see the meaning of this result in terms of observable quantities.

Let us first consider the (approximately) composition-independent deviations from general relativity, i.e. those that consist in violations of the "strong"

[1]See, e.g., the review of global cosmological parameters (chap. 17) in [Groom *et al.*, 2000]

equivalence principle. As reviewed in Section 1.2, most composition-independent gravitational experiments (in the solar system or in binary pulsars) consider the long-range interaction between objects whose masses are essentially baryonic (the Sun, planets, neutron stars). As argued in Section 1.3, [equation (1.94)] the relevant coupling coefficient α_A is then approximately universal and given by the logarithmic derivative of the QCD confinement scale $\Lambda_{\rm QCD}(\varphi)$, because the mass of hadrons is essentially given by a pure number times $\Lambda_{\rm QCD}(\varphi)$. [We shall consider below the small, non-universal, corrections to $m_A(\varphi)$ and $\alpha_A(\varphi)$ linked to QED effects and quark masses.] Remembering from equation (3.1) the fact that, in the string frame (where there is a fixed cut-off linked to the string mass $\widetilde{M}_s \sim (\alpha')^{-1/2}$) the gauge coupling is dilaton-dependent ($g_F^{-2} = B_F(\varphi)$), we see that (after conformal transformation) the Einstein-frame confinement scale has a dilaton-dependence of the form

$$\Lambda_{\rm QCD}(\varphi) \sim C^{-1/2}\, B_g^{-1/2}(\varphi) \exp[-8\pi^2\, b_3^{-1}\, B_F(\varphi)]\, \widetilde{M}_s\,, \tag{3.41}$$

where b_3 denotes the one-loop (rational) coefficient entering the Renormalization Group running of g_F. Here $B_F(\varphi)$ denotes the coupling to the SU(3) gauge fields. For simplicity, we shall assume that (modulo rational coefficients) all gauge fields couple (near the string cut off) to the same $B_F(\varphi)$. This yields the following approximately universal dilaton coupling to hadronic matter

$$\alpha_{\rm had}(\varphi) \simeq \left[\ln\left(\frac{\widetilde{M}_s}{\Lambda_{\rm QCD}}\right) + \frac{1}{2}\right] \frac{\partial \ln B_F^{-1}(\varphi)}{\partial\, \varphi}\,. \tag{3.42}$$

We recall that the quantity $\alpha_{\rm had}(\varphi)$, which measures the coupling of the dilaton to hadronic matter, should not be confused with any "strong" gauge coupling, $\alpha_s = g_s^2/4\pi$. Numerically, the coefficient in front of the R.H.S. of (3.42) is of order 40. Consistently with our basic assumption (3.3), we parametrize the φ dependence of the gauge coupling $g_F^2 = B_F^{-1}$ as

$$B_F^{-1}(\varphi) = B_F^{-1}(+\infty)\,[1 - b_F\, e^{-c\varphi}]\,. \tag{3.43}$$

Note that, like b_λ (see section 3.1.1), also b_F is expected to be smallish [$\sim B_F^{-1}(+\infty)$ or, equivalently, $\sim C_F^{-1}$ in the notations of (3.3)] and typically the ratio b_F/b_λ is of order unity. We finally obtain

$$\alpha_{\rm had}(\varphi) \simeq 40\, b_F\, c\, e^{-c\varphi}\,. \tag{3.44}$$

We can now insert the estimate (3.28) of the value of φ reached because of the cosmological evolution. Neglecting the $\mathcal{O}(1)$ renormalization factor due to quantum noise, we get the estimate

$$\alpha_{\rm had}(\varphi_{\rm end}) \simeq 3.2\, \frac{b_F}{b_\lambda\, c}\, \delta_H^{\frac{4}{n+2}}\,, \tag{3.45}$$

$$\alpha_{\rm had}^2(\varphi_{\rm end}) \simeq 10 \left(\frac{b_F}{b_\lambda\, c}\right)^2 \delta_H^{\frac{8}{n+2}}\,. \tag{3.46}$$

As said above, it is plausible to expect that the quantity c (which is a ratio) and the ratio b_F/b_λ are both of order unity. This then leads to the numerical estimate $\alpha_{\rm had}^2 \sim 10\,\delta_H^{\frac{8}{n+2}}$, with $\delta_H \simeq 5\times 10^{-5}$. An interesting aspect of this result is that the expected present value of $\alpha_{\rm had}^2$ depends rather strongly on the value of the exponent n (which entered the inflaton potential $V(\chi)\propto\chi^n$). In the case $n=2$ (i.e. $V(\chi)=\frac{1}{2}\,m_\chi^2\,\chi^2$) we have $\alpha_{\rm had}^2\sim 2.5\times 10^{-8}$, while if $n=4$ ($V(\chi)=\frac{1}{4}\,\lambda\,\chi^4$) we have $\alpha_{\rm had}^2\sim 1.8\times 10^{-5}$.

These numbers satisfy the present experimental limits on SEP violations reviewed in Section 1.2. Concerning solar-system (post-Newtonian) tests it has been shown that the two main "Eddington" parameters $\gamma-1$ and $\beta-1$ measuring post-Newtonian deviations from general relativity are linked to the dilaton coupling $\alpha_{\rm had}(\varphi)$ by equations (1.57) and (1.58):

$$\gamma-1=-2\,\frac{\alpha_{\rm had}^2}{1+\alpha_{\rm had}^2}\simeq -2\,\alpha_{\rm had}^2\,, \tag{3.47}$$

$$\beta-1=\frac{1}{2}\,\frac{\alpha'_{\rm had}\,\alpha_{\rm had}^2}{(1+\alpha_{\rm had}^2)^2}\simeq\frac{1}{2}\,\alpha'_{\rm had}\,\alpha_{\rm had}^2\,, \tag{3.48}$$

where $\alpha'_{\rm had}\equiv\partial\alpha_{\rm had}(\varphi)/\partial\varphi$.

From equation (3.44) we see that $\alpha'_{\rm had}\simeq -c\,\alpha_{\rm had}$, so that the deviation $\beta-1$ is $\mathcal{O}(\alpha_{\rm had}^3)$ and thereby predicted to be too small to be phenomenologically interesting. This leaves $\gamma-1\simeq -2\,\alpha_{\rm had}^2$ as the leading observable deviation. We have seen that the best current solar-system limit on $\gamma-1$ comes from Very Long Baseline Interferometry measurements of the deflection of radio waves by the Sun and is (approximately) $|\gamma-1|\lesssim 2\times 10^{-4}$, corresponding to $\alpha_{\rm had}^2\lesssim 10^{-4}$ [equation (1.70)]. In addition to solar-system tests, we should also consider binary-pulsar tests which provide another high-precision window on possible deviations from general relativity. They have been analyzed in terms of the two quantities $\alpha_{\rm had}$ (denoted α) and $\alpha'_{\rm had}$ (denoted β) in [Damour and Esposito-Farèse, 1998]. The final conclusion is that the binary-pulsar limit on $\alpha_{\rm had}$ is of order $\alpha_{\rm had}^2\lesssim 10^{-3}$.

At this stage it seems that the runaway scenario explored here is leading to deviations from general relativity which are much smaller than present experimental limits. However, we must turn our attention to *composition-dependent* effects which turn out to be much more sensitive tests.

3.2.2 Composition-dependent deviations from general relativity

Let us then consider situations where the non-universal couplings of the dilaton induce (apparent) violations of the equivalence principle. Let us start by considering the composition-dependence of the dilaton coupling α_A, equation (3.30), i.e. the dependence of α_A on the type of matter we consider. As we saw in Section 1.3, at the Newtonian approximation the interaction potential between particle A and particle B is $-G_{AB}\,m_A\,m_B/r_{AB}$ where [Damour and Polyakov, 1994]

$$G_{AB}=G(1+\alpha_A\,\alpha_B)\,. \tag{3.49}$$

Here, G is the bare gravitational coupling constant entering the Einstein-frame action (3.7), and $\alpha_A = \alpha_A(\varphi)$ is the strength of the dilaton coupling to A-particles, taken at the present (cosmologically determined) VEV of φ. The term $\alpha_A\,\alpha_B$ comes from the additional attractive effect of dilaton exchange. Two test masses, made respectively of A- and B-type particles will then fall in the gravitational field generated by an external mass m_E with accelerations differing by

$$\left(\frac{\Delta a}{a}\right)_{AB} \equiv 2\,\frac{a_A - a_B}{a_A + a_B} \simeq (\alpha_A - \alpha_B)\,\alpha_E\,. \tag{3.50}$$

We have seen above that in lowest approximation $\alpha_A \simeq \alpha_{\rm had}$ does not depend on the composition of A. We need, however, now to retain the small composition-dependent effects to α_A linked to the φ-dependence of QED and quark contributions to m_A. This has been investigated in [Damour and Polyakov, 1994] with the result

$$\left(\frac{\Delta a}{a}\right)_{AB} = \left(\frac{\alpha_{\rm had}}{40}\right)^2 \left[C_B\,\Delta\left(\frac{B}{M}\right) + C_D\,\Delta\left(\frac{D}{M}\right) + C_E\,\Delta\left(\frac{E}{M}\right)\right]_{AB}, \tag{3.51}$$

where $(\Delta X)_{AB} \equiv X_A - X_B$, where $B \equiv N+Z$ is the baryon number, $D \equiv N-Z$ the neutron excess, $E \equiv Z(Z-1)/(N+Z)^{1/3}$ a quantity linked to nuclear Coulomb effects, and where $M \equiv m/u$ denotes the mass in atomic mass unit, $u = 931.49432$ MeV. It is difficult (and model-dependent) to try to estimate the coefficients C_B and C_D. It was argued in [Damour and Polyakov, 1994] that their contributions to (3.51) is generically expected to be sub-dominant with respect to the last contribution, $\propto C_E$, which can be better estimated because it is linked to the φ-dependence of the fine-structure constant $e^2 \propto B_F^{-1}(\varphi)$. This then leads to the numerical estimate $C_E \simeq 3.14 \times 10^{-2}$ and a violation of the universality of free fall approximately given by

$$\left(\frac{\Delta a}{a}\right)_{AB} \simeq 2 \times 10^{-5}\,\alpha_{\rm had}^2 \left[\left(\frac{E}{M}\right)_A - \left(\frac{E}{M}\right)_B\right]. \tag{3.52}$$

The values of B/M, D/M and E/M have been computed in [Damour, 1996*b*]. For mass-pairs that have been actually used in recent experiments (such as Beryllium and Copper), as well as for mass-pairs that are planned to be used in forthcoming experiments (such as Platinum and Titanium) one finds: $(E/M)_{\rm Cu} - (E/M)_{\rm Be} = 2.56$, $(E/M)_{\rm Pt} - (E/M)_{\rm Ti} = 2.65$. Using the average estimate $\Delta(E/M) \simeq 2.6$, we get from (3.52) and (3.46) the estimate

$$\left(\frac{\Delta a}{a}\right) \simeq 5.2 \times 10^{-5}\,\alpha_{\rm had}^2 \simeq 5.2 \times 10^{-4} \left(\frac{b_F}{b_\lambda\,c}\right)^2 \delta_H^{\frac{8}{n+2}}\,. \tag{3.53}$$

Note also (from (3.47)) the link between composition-dependent effects and post-Newtonian ones

$$\left(\frac{\Delta a}{a}\right) \simeq -2.6 \times 10^{-5}(\gamma - 1)\,. \tag{3.54}$$

As current tests of the universality of free fall (UFF) have put limits in the 10^{-12} range (e.g. $(\Delta a/a)_{\rm Be\,Cu} = (-1.9 \pm 2.5) \times 10^{-12}$ from [Su *et sl.*, 1994]), we see from equation (3.54) that this corresponds to limits on $\gamma - 1$ or $\alpha_{\rm had}^2$ in the 10^{-7} range. Therefore tests of the UFF put much more stringent limits on dilaton models than solar-system or binary-pulsar tests.

If we insert the estimate $\delta_H \sim 5\times10^{-5}$ in (3.53) we obtain a level of violation of UFF due to a runaway dilaton which is

$$\frac{\Delta a}{a} \simeq 1.3 \left(\frac{b_F}{b_\lambda\, c}\right)^2 \times 10^{-12} \quad \text{for } n = 2\,, \tag{3.55}$$

$$\frac{\Delta a}{a} \simeq 0.98 \left(\frac{b_F}{b_\lambda\, c}\right)^2 \times 10^{-9} \quad \text{for } n = 4\,. \tag{3.56}$$

At face value, one is tempted to conclude that a scenario with $n = 4$ (i.e. $V(\chi) \propto \chi^4$) tends to be too weak an attractor towards $\varphi = +\infty$ to be naturally compatible with equivalence-principle tests. [See, however, the discussion below.] On the other hand, the simple scenario $n = 2$ ($V(\chi) = \frac{1}{2}\, m_\chi^2\, \chi^2$) is quite appealing in that it naturally provides enough attraction towards $\varphi = +\infty$ to be compatible with all existing experimental tests. At the same time it suggests that a modest improvement in the precision of UFF experiments might discover a violation caused by a runaway dilaton.

3.2.3 Cosmological variation of "constants"

Let us now consider another possible deviation from General Relativity and the standard model: a possible variation of the coupling constants, most notably of the fine structure constant $e^2/\hbar c$ on which the strongest limits are available. We will discuss first the effects due to the cosmological time-variation of the homogeneous component of φ and, in the next subsection, the possible spatial (and time) variations due to quantum fluctuations of φ as they got amplified during inflation.

Consistently with our previous assumptions we expect $e^2 \propto B_F^{-1}(\varphi)$ so that, from (3.43),

$$e^2(\varphi) = e^2(+\infty)\,[1 - b_F\, e^{-c\varphi}]\,. \tag{3.57}$$

The present logarithmic variation of e^2 (using again $dp = H\,dt$; $\varphi' = d\varphi/dp$) is thus given by

$$\frac{d\ln e^2}{H\,dt} = \frac{d\ln e^2}{dp} \simeq b_F\, c\, e^{-c\varphi}\, \varphi'_0\,, \tag{3.58}$$

where the current value of φ', φ'_0, is given in general by equation (3.35). Using equation (3.44), we can rewrite the result (3.58) in terms of the hadronic coupling:

$$\frac{d\ln e^2}{H\,dt} \simeq \frac{1}{40}\alpha_{\rm had}\varphi'_0\,. \tag{3.59}$$

As said in section 3.1.2, we have basically two alternatives concerning the current coupling of the dilaton to the dominant energy sources in the universe. These two alternatives lead to drastically different predictions for the current value of the rate of variation of the fine-structure constant. We shall consider these two alternatives in turn.

In the conservative case where the dilaton does not play any special role in the present accelerated phase of the universe ($\alpha_V \simeq 0$) nor does it have any stronger coupling to dark matter than to visible matter ($\alpha_m \simeq -b_m\, c\, e^{-c\varphi}$) the dilaton "velocity" φ' is exponentially suppressed (so that, from (3.37), $\Omega_V \simeq 1 - \Omega_m$) and by equation (3.35) one obtains

$$\frac{d\ln e^2}{H\,dt} \simeq -\frac{\Omega_m}{\Omega_m + 2\Omega_V}\, b_F\, c\, e^{-c\varphi} \alpha_m(\varphi) \simeq \frac{\Omega_m}{2-\Omega_m}\, b_F\, b_m\, c^2\, e^{-2c\varphi}. \tag{3.60}$$

An indicative value for the ratio $\Omega_m/(\Omega_m + 2\Omega_V) \simeq \Omega_m/(2-\Omega_m)$, by taking for instance $\Omega_m = 0.3$, is 0.18. As above, it is useful to relate (3.60) to the estimate (3.44) for $\alpha_{\rm had}$. This yields

$$\frac{d\ln e^2}{H\,dt} \simeq \frac{1}{(40)^2}\,\frac{\Omega_m}{2-\Omega_m}\,\frac{b_m}{b_F}\,\alpha^2_{\rm had}\,. \tag{3.61}$$

In terms of the UFF level $\Delta a/a$ predicted by our model in (3.53) we see also that

$$\frac{d\ln e^2}{H\,dt} \simeq 12\,\frac{\Omega_m}{2-\Omega_m}\,\frac{b_m}{b_F}\,\frac{\Delta a}{a}\,. \tag{3.62}$$

Even if the universe were completely dominated by dark matter ($\Omega_m = 1$) we see, assuming that b_m/b_F is of order unity, that current experimental limits on UFF ($\Delta a/a \lesssim 10^{-12}$) imply (within dilaton models) that $|d\ln e^2/dt| \lesssim 10^{-11}\,H \sim 10^{-21}\,{\rm yr}^{-1}$ (the sign of $d\ln e^2/dt$ being given by the sign of b_m/b_F). This level of variation is much smaller than the current best limit on the time variation of e^2, namely $|d\ln e^2/dt| \lesssim 5\times 10^{-17}\,{\rm yr}^{-1} \sim 5\times 10^{-7}\,H$, as obtained from an analysis of Oklo data [Shlyakhter, 1976], [Damour and F. Dyson, 1996]. (Note that the assumption-dependent analysis of Ref. [Olive *et al.*, 2002] gives a limit on the variation of e^2 which is strengthened by about two orders of magnitude.)

The situation, however, is drastically different if we consider the alternative case where the dilaton coupling to the current dominant energy sources does not tend to triviality, as in the case of a φ-dependent vacuum energy $V(\varphi) = V_0 + V_1 e^{-c\varphi}$ when the first term is zero or negligible. In such a case the dilaton shares a relevant part of the total energy density and more significant (though still quite constrained by UFF data) variations of the coupling constants are generally expected. A general expression for the dilaton "velocity" is given in eq. (3.39) in terms of observable quantities. Using equations (3.39) and (3.59) one can relate the expected variation of the electromagnetic coupling constant to the hadronic coupling:

$$\frac{d\ln e^2}{H\,dt} \simeq \pm\,\frac{\alpha_{\rm had}}{40}\,\sqrt{1+q_0-3\Omega_m/2}\,. \tag{3.63}$$

We can also use the estimate (3.45) relating $\alpha_{\rm had}$ to the density fluctuations generated during inflation. We obtain

$$\frac{d\ln e^2}{H\,dt} \simeq \pm 8\times 10^{-2}\,\sqrt{1+q_0-3\Omega_m/2}\,\frac{b_F}{b_\lambda c}\,\delta_H^{\frac{4}{n+2}}\,. \tag{3.64}$$

However, in view of the theoretical uncertainties attached to the initial conditions $\chi_{\rm in}$ and $\varphi_{\rm in}$ used in the estimate (3.45), as well as the ones associated to the order unity ratio $b_F/(b_\lambda c)$, it is more interesting to rewrite our prediction in terms of *observable* quantities. Using again the link equation (3.53) between $\alpha_{\rm had}$ and the observable violation of the universality of free fall (UFF) the above result can be written in the form

$$\frac{d\ln e^2}{H\,dt} \simeq \pm 3.5\times 10^{-6}\,\sqrt{1+q_0-3\Omega_m/2}\,\sqrt{10^{12}\frac{\Delta a}{a}}\,. \tag{3.65}$$

Note that the sign of the variation of e^2 is in general model-dependent (as it depends both on the sign of b_F and the sign of φ_0'). Specific classes of models might, however, favour particular signs of de^2/dt. For instance, from the point of view of [Veneziano, 2002] one would expect the $\mathcal{O}(e^{-\phi})$ terms in equation (3.3) to be positive, which would then imply that b_F is positive. If we combine this information with the prediction equation (3.36) of the model [Gasperini, Piazza and Veneziano, 2002] implying that φ' is also positive, we would reach the conclusion that e^2 must be currently *increasing*.

Independently of this question of the sign, we see that equation (3.65) predicts an interesting link between the observational violation of the UFF (constrained to $\Delta a/a \lesssim 10^{-12}$), and the current time-variation of the fine-structure constant. Contrary to the relation (3.62), obtained above under the alternative assumption about the dilaton dependence of the dominant cosmological energy, which predicted a relation linear in $\Delta a/a$, we have here a relation involving the square root of the UFF violation (such a relation is similar to the result of [Damour, Gibbons and Gundlach, 1990] which concerned the time-variation of the Newton constant).

The phenomenologically interesting consequence of equation (3.65) is to predict a time-variation of constants which may be large enough to be detected by high-precision laboratory experiments. Indeed, using $H_0 \simeq 66$ km/s/Mpc, and the plausible estimates $\Omega_m = 0.3$, $q_0 = -0.4$, equation (3.65) yields the numerical estimate $d\ln e^2/dt \sim \pm 0.9\times 10^{-16}\,\sqrt{10^{12}\Delta a/a}\,{\rm yr}^{-1}$. Therefore, the current bound on UFF violations ($\Delta a/a \sim 10^{-12}$) corresponds to the level $10^{-16}{\rm yr}^{-1}$, which is comparable to the planned sensitivity of currently developed cold-atom clocks [Salomon *et al.*, 2001]. [Present laboratory bounds are at the $10^{-14}{\rm yr}^{-1}$ level [Prestage, Tjoelker, and Maleki, 1995, Salomon *et al.*, 2001].] Note that if we insert in equation (3.65) the secure bounds $\Omega_m > 0.2$ and $q_0 < 0$ (leading to the limit equation (3.40)), we get as maximal estimate of the time variation of the fine-structure constant $d\ln e^2/dt \sim \pm 2.0\times 10^{-16}\,\sqrt{10^{12}\Delta a/a}\,{\rm yr}^{-1}$. We note also that the upper limit on the variation of e^2 given by the Oklo data, i.e. $|d\ln e^2/dt| \lesssim 5\times 10^{-17}\,{\rm yr}^{-1}$ [Shlyakhter, 1976], [Damour and F. Dyson, 1996], "corresponds" to a violation of the UFF at the level $\sim 10^{-13}$.

In this respect, it is interesting to consider not only the *present* variation of e^2 (the only one relevant for laboratory experiments), but also its variation over several billions of years. (We recall that the Oklo phenomenon took place about two billion years ago, and that astronomical observations constrain the variation of e^2 over the last ten billion years or so). In particular, an interesting question is to see whether our model could reconcile the Oklo limit (which corresponds to a redshift $z \simeq 0.14$) with the recent claim [Webb *et al.*, 2001] of a variation $\Delta e^2/e^2 = (-0.72 \pm 0.18) \times 10^{-5}$ around redshifts $z \approx 0.5 - 3.5$ as proposed in [Sandvik, Barrow and Magueijo, 2002, Olive and Pospelov, 2002]. The only hope of reconciling the two results would be to allow for a faster variation of e^2 for redshifts $z > 0.5$. Such recent redshifts have (apparently) been connected to a transition from matter dominance to vacuum dominance. Let us see whether taking into account this transition might allow for a large enough change of e^2 around redshifts $z \approx 0.5 - 3.5$. We must clearly assume the "strong coupling" scenario $\alpha_m = \mathcal{O}(1)$. In this scenario, the variation of φ during the matter era is given by equation(3.34). Neglecting, for simplicity, the transient evolution effects localized around the matter-vacuum transition (and treating both $\varphi'_m = -\alpha_m$ and $\varphi'_V = \varphi'_0$ as constants), the solution giving the recent cosmological evolution of φ reads $\varphi - \varphi_0 = -\varphi'_0 \ln(1+z)$ during the vacuum era, and $\varphi - \varphi_0 = -\varphi'_0 \ln(1+z_*) - \varphi'_m \ln[(1+z)/(1+z_*)]$ during the matter era (the index 0 refers to the present epoch, i.e. $z = 0$; z_* denotes the transition redshift). Inserting this change in equation(3.57) leads to the following expression for the cosmological change of the fine-structure constant:

$$\frac{e^2 - e_0^2}{e_0^2} = -\mathrm{sign}(b_F)\, 3.5 \times 10^{-6} \left[\varphi'_0 \ln(1+z_*) + \varphi'_m \ln \frac{1+z}{1+z_*}\right] \sqrt{10^{12} \frac{\Delta a}{a}}\,. \tag{3.66}$$

Here, we have written the result for the matter era. During the vacuum era the bracket is simply $[\varphi'_0 \ln(1+z)]$. Remembering that the absolute value of φ'_m is (like that of φ'_V) observationally constrained to be smaller than $\sqrt{0.3} \simeq 0.55$ (and that φ'_0 is also constrained by $|\varphi'_V| < 0.84$), we see that there is no way, within our model, to explain a variation of e^2 as large as $\Delta e^2/e^2 = (-0.72 \pm 0.18) \times 10^{-5}$ around redshifts $z \approx 0.5 - 3.5$ [Webb *et al.*, 2001]. In our model, even under the assumption that UFF is violated just below the currently tested level, such a change would have to correspond to a value $|\varphi'_m| > 2$, entailing observationally unacceptable modifications of standard cosmology. [For instance, in the model [Gasperini, Piazza and Veneziano, 2002] a value as large as $\alpha_m > 1$ already leads to a pathological behaviour ("total dragging") where all the components scale like radiation.] This difficulty of reconciling the Oklo limit with the claim of [Webb *et al.*, 2001] was addressed in [Olive and Pospelov, 2002, Sandvik, Barrow and Magueijo, 2002] within a different class of models, namely with a field ϕ which *does not* couple universally to all gauge fields $F_{\mu\nu}$, as the dilaton φ is expected to do. The fact that the field ϕ in [Olive and Pospelov, 2002] (or ψ in [Sandvik, Barrow and Magueijo, 2002]) is assumed to couple only to the electromagnetic gauge field drastically changes our equation (3.53) and allows one to satisfy the UFF limit $\Delta a/a \lesssim 10^{-12}$, for a stronger coupling of ϕ to electromagnetism than in our class of models, i.e. (in our notation) for a larger $d\ln B_F(\varphi)/d\varphi$. This explains why Ref. [Olive and Pospelov, 2002] could construct some explicit (but fine-tuned) models in which all observational limits (UFF, Oklo,...) could be met and still allow for a variation of e^2 as strong as the claim[23]. The maximal variation predicted by equation(3.66) for redshifts corresponding to the matter era (obtained when $\Delta a/a = 10^{-12}$ and $\varphi'_m = \pm\sqrt{0.3}$; and assuming a smaller value of φ'_0 to be compatible with the Oklo constraint), is of order $\Delta e^2/e^2 = \pm 1.9 \times 10^{-6}$. This is only a factor ~ 4 below the claim [Webb *et al.*, 2001] and is at the level of their one sigma error bar. Therefore a modest improvement in the observational precision (accompanied by an improved control of systematics) will start to probe a domain of variation of constants which, according to our scenario, corresponds to an UFF violation smaller than the 10^{-12} level.

3.2.4 Spatio-temporal fluctuations of the "constants"

We now turn to the second possible source of spatial-temporal variations for e^2 in our model, the quantum fluctuations of the dilaton generated during inflation. Within linear perturbation theory, the relevant calculation may be summarized as follows.

Consider a flat FRW universe $ds^2 = -dt^2 + a(t)^2\,\Sigma_i dx_i^2$. The dilaton fluctuations can be expanded in Fourier components $\delta\varphi_{\mathbf{k}}$ of given comoving momentum $\mathbf{k}$ as follows:

$$\delta\varphi(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}} \int d^3k\, \delta\varphi_{\mathbf{k}}(t) e^{i\mathbf{kx}}, \tag{3.67}$$

where t is the cosmological time. Each Fourier mode $\delta\varphi_{\mathbf{k}}$ "leaves" the horizon during inflation with an amplitude $\sim \widehat{H}_{\rm ex}(k)/\sqrt{2k^3}$ [Liddle and Lyth, 2000] where, by definition, $\widehat{H}_{\rm ex}(k)$ is the value of the dimensionless Hubble expansion rate as ka^{-1} equals H during inflation (note that we denote here $\widehat{H}_{\rm ex}$ what was denoted $\widehat{H}_\times$ above). Well after the exit ($k \ll aH$) the amplitude of each mode "freezes out", i.e. remains roughly constant, until it reenters the horizon during the post-inflationary epoch ($ka_{re}^{-1} \simeq H_{re}$). After re-entry the amplitude starts to damp out as a^{-1}. For a given Fourier mode $\delta\varphi_{\mathbf{k}}(t)$, the latter damping effect is described by the piecewise function

$$f_z(k) \equiv \begin{cases} 1 & \text{if} \quad a_0^{-1}H_0^{-1}\,k < (z+1)^{1/2} \\ a_0^2H_0^2\,(z+1)k^{-2} & \text{if} \quad (z+1)^{1/2} < a_0^{-1}H_0^{-1}\,k < 10^2 \\ 10^{-2}a_0H_0\,(z+1)k^{-1} & \text{if} \quad a_0^{-1}H_0^{-1}\,k > 10^2 \end{cases} \tag{3.68}$$

Here the cosmological redshift $z = a_0/a(t) - 1$ has been introduced in replacement of the cosmological time t. The first case refers to Fourier modes that have not reentered yet at redshift z and whose amplitudes are still frozen. The second and third cases refer to modes that reenter during matter and radiation domination respectively. Putting all together, and assuming a gaussian probability distribution for the perturbations, we have:

$$\langle \delta\varphi_{\mathbf{k}}(t)^* \, \delta\varphi_{\mathbf{k}'}(t')\rangle \;=\; \frac{\widehat{H}^2_{\rm ex}(k)}{2k^3}\, f_z(k)\, f_{z'}(k)\, \delta^3(\mathbf{k}-\mathbf{k}') \,. \tag{3.69}$$

Possible spatial/temporal variations of e^2 induced by the fluctuations of the dilaton will be given by

$$\left.\frac{\Delta^{\rm fluc} e^2}{e^2}\right|_{(\mathbf{x},t;\,\mathbf{x}',t')} \;=\; \frac{d\ln e^2}{d\varphi}\, \Delta^{\rm fluc}\varphi|_{(\mathbf{x},t;\,\mathbf{x}',t')} \,, \tag{3.70}$$

where the r.m.s $\Delta^{\rm fluc}\varphi$ between two events $(\mathbf{x},t)$ and $(\mathbf{x}',t')$ is defined as follows

$$\begin{aligned}
\Delta^{\rm fluc}\varphi|^2_{(\mathbf{x},t;\,\mathbf{x}',t')} \;&\equiv \langle\, [\delta\varphi(\mathbf{x},t) - \delta\varphi(\mathbf{x}',t')]^2 \rangle \\
&= \frac{\widehat{H}^2_{\rm ex}}{(2\pi)^3} \int \frac{d^3k}{2\,k^3}\, \left[f_z(k)^2 + f_{z'}(k)^2 - 2 f_z(k) f_{z'}(k)\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \right] \qquad (3.71)\\
&= \frac{\widehat{H}^2_{\rm ex}}{(2\pi)^2} \int_0^\infty \frac{dk}{k} \left\{ [f_z(k) - f_{z'}(k)]^2 + 2 f_z(k) f_{z'}(k) \left[1 - \frac{\sin kx}{kx} \right] \right\} .
\end{aligned}$$

Here, $x \equiv |\mathbf{x}-\mathbf{x}'|$ is the coordinate distance between the two events and, consistently with the slow-roll approximation, the Hubble expansion rate at exit has been assumed to be scale-invariant: $\widehat{H}_{\rm ex}(k) \simeq \widehat{H}_{\rm ex} \simeq 3\times 10^{-5}$.

If one considers spatial fluctuations over terrestrial or solar system proper length scales $l = a_0 k^{-1} \ll H_0^{-1}$ at the present time $t = t' = t_0$, the first square brackets in (3.71) vanishes and one can expand the sine function at small kx obtaining

$$\Delta^{\rm fluc}\varphi|_{l;\,z=0} \;\simeq\; \frac{\widehat{H}_{\rm ex}}{2\pi}\,\frac{H_0\, l}{\sqrt{3}}; \qquad\qquad \left.\frac{\Delta^{\rm fluc} e^2}{e^2}\right|_{l;\,z=0} \;\simeq\; 10^{-2}\, \alpha_{\rm had}\, \widehat{H}_{\rm ex}\, H_0\, l \,. \tag{3.72}$$

As expected, these variations are extremely small, $\Delta^{\rm fluc} e^2/e^2|_{l;\,z=0} \simeq 10^{-33}\, l/$km. It is also interesting to compare dilaton fluctuations at different redshifts along a comoving observer wordline. By putting $x \simeq 0$ in (3.71) the second term in the square brackets vanishes and one has:

$$\begin{aligned}
\Delta^{\rm fluc}\varphi|_{z;\,x=0} \;&\simeq\; \frac{\widehat{H}_{\rm ex}}{2\pi}\,\frac{1}{\sqrt{2}} \left[\log(1+z) - \frac{z}{1+z} + \frac{10^{-8}}{2} z^2 \right]^{1/2} \\
&\simeq\; \frac{\widehat{H}_{\rm ex}}{2\pi} \left[\frac{z}{2} - \frac{z^2}{3} + \ldots \right] .
\end{aligned} \tag{3.73}$$

It is slightly more complicated to compare dilaton fluctuations between "now" and events at redshift z along a null ray. Expanding in powers of z around $z = 0$ one gets from (3.71), after a straightforward calculation:

$$\Delta^{\rm fluc}\varphi|_z \;\simeq\; \frac{\widehat{H}_{\rm ex}}{2\pi} \left[\frac{1}{2}\sqrt{\frac{7}{3}}\, z - \frac{2}{\sqrt{21}}\, z^2 + \ldots \right] . \tag{3.74}$$

Numerically, at redshift $z \sim 1$, the effects of dilatonic fluctuations are given by $\Delta^{\rm fluc}\varphi|_{z=1} \sim \widehat{H}_{\rm ex}/(2\pi) \sim 5 \times 10^{-6}$. This is to be contrasted with the effects of the cosmic, homogeneous evolution which yields $\Delta\varphi|_{z=1} \simeq \alpha_m$. In the "normal" case where $\alpha_m \sim e^{-c\varphi} \sim \delta_H^{4/(n+2)}$, the two effects, though a priori unrelated, are related in our scenario , when $n = 2$. Indeed, if $n = 2$, $\delta_H^{4/(n+2)} = \delta_H \sim 5 \times 10^{-5}$ is linked to $\widehat{H}_{\rm ex}/(2\pi)$ via $\delta(\chi_{\rm ex}) = A\widehat{H}_{\rm ex}/(2\pi)$ with $A = (8/3)V/\partial_\chi V = (8/3)(\chi/n) \simeq 40/(3\sqrt{n}) \sim 10$. On the other hand, in the case where φ is strongly coupled to dark matter, the homogeneous evolution $\Delta\varphi|_{z=1} \simeq \alpha_m \sim 1$ is parametrically larger than the fluctuations $\Delta^{\rm fluc}\varphi|_{z=1} \sim \widehat{H}_{\rm ex}/(2\pi)$.

To conclude on this subsection, we see that the inhomogeneous space-time fluctuations of the fine-structure constant are typically too small to be observable (if the limits from UFF are already satisfied), being suppressed, relative to their natural values $H_0 l$, $H_0 t$, by the small factor $\alpha_{\rm had}\widehat{H}_{\rm ex}$.

3.3 Summary and conclusion

We have studied the dilaton-fixing mechanism of [Damour and Polyakov, 1994] within the context where the dilaton-dependent low-energy couplings are extremized at $\varphi = +\infty$, i.e. for infinitely large values of the bare string coupling $g_s^2 = e^\phi \simeq e^{c\varphi}$. [The crucial coupling to the inflaton, say $\lambda(\varphi)$ in equation(3.14), must be *minimized* at $\varphi \to +\infty$; the other couplings can be either minimized or maximized there.] This possibility of a fixed point at infinity (in bare string coupling space) has been recently suggested [Veneziano, 2002], and its late-cosmological consequences have been explored in [Gasperini, Piazza and Veneziano, 2002]. We found that a primordial inflationary stage, with inflaton potential $V(\chi) = \lambda(\varphi)\,\chi^n/n$, was much less efficient in decoupling a dilaton with least couplings at infinity than in the case where the least couplings are reached at a finite value of φ (as in [Damour and Polyakov, 1994, Damour and Vilenkin, 1996]). This reduced efficiency has interesting phenomenological consequences. Indeed, it predicts much larger observable deviations from general relativity. In the case of the simplest chaotic potential [Linde, 1990] $V(\chi) = \frac{1}{2}\,m_\chi^2(\varphi)\,\chi^2$, we find that, under the simplest assumptions about the pre-inflationary state, this scenario predicts violations of the universality of free fall (UFF) of order $\Delta a/a \sim 5 \times 10^{-4}\,\delta_H^2$ where δ_H is the density fluctuation generated by inflation on horizon scales. The observed level of large-scale density (and cosmic microwave background temperature) fluctuations fixes δ_H to be around 5×10^{-5} which finally leads to a prediction for a violation of the UFF near the $\Delta a/a \sim 10^{-12}$ level. This is naturally compatible with present experimental tests of the equivalence principle, and suggests that a modest improvement in the precision of UFF tests might be able to detect a deviation linked to dilaton exchange with a coupling reduced by the attraction towards the fixed point at infinity. Because of the presence of unknown dimensionless ratios $(c, b_F/b_\lambda)$ in our estimates, and of quantum noise in the evolution of the dilaton, we cannot give sharp quantitative estimates of $\Delta a/a$. However, we note that dilaton-induced violations of the UFF have a rather precise signature with a composition-dependence of the form (3.51), with probable domination by the last (Coulomb energy) term [Damour and Polyakov, 1994]. As explored in [Damour, 1996*b*] this signature is quite distinct from UFF violations induced by other fields, such as a vector field. We note that the approved Centre National d'Etudes Spatiales (CNES) mission MICROSCOPE [P. Touboul *et al.*, 2001] (to fly in 2004) will explore the level $\Delta a/a \sim 10^{-15}$, while the planned National Aeronautics and Space Agency (NASA) and European Space Agency (ESA) mission STEP (Satellite Test of the Equivalence Principle)[P. W. Worden, 1996] could explore the $\Delta a/a \sim 10^{-18}$ level. Our scenario gives additional motivation for such experiments and suggests that they might find a rather strong violation signal, whose composition-dependence might then be studied in detail to compare it with equation (3.51).

In the case of inflationary potentials $V(\chi) \propto \chi^n$ with $n > 2$ our simplest estimates predict a violation of the UFF of order $\Delta a/a \sim 5 \times 10^{-4}\, \delta_H^{\frac{8}{n+2}}$ which is larger than 10^{-12}. At face value this suggests that existing UFF experimental data can be interpreted as favouring $n \leq 2$ over $n > 2$. However, we must remember that our estimates have made several simplifying assumptions. It is possible that the large quantum fluctuations of the inflaton in the self-regenerating regime $\chi > \chi_{\rm in}$, with $\chi_{\rm in}$ defined by equation (3.21), can give more time for φ to run away towards large values, so that the effective value of $e^{c\varphi_{\rm in}}$ to be used in equation (3.24) turn out to dominate the first term in the R.H.S. that we have used for our estimates. We leave to future work a study of the system of Langevin equations describing the coupled fluctuations of ϕ and χ during the self-regenerating regime.

Finally let us note some other conclusions of our work.

We recover the conclusion of previous works on dilaton models that the most interesting experimental probes of a massless weakly coupled dilaton are tests of the UFF. The composition-independent gravitational tests (solar-system, binary-pulsar) tend to be much less sensitive probes (as highlighted by the relations (3.54), (3.61) and (3.62)).

However, a possible exception concerns the time-variation of the coupling constants. Here the conclusion depends crucially on the assumptions made about the couplings of the dilaton to the cosmologically dominant forms of energy (dark matter and/or dark energy). If these couplings are of order unity (and as large as is phenomenologically acceptable, i.e. so that $(\varphi_0')^2 = 0.7$), the present time variation of the fine-structure constant is linked to the violation of the UFF by the relation $d\ln e^2/dt \sim \pm 2.0 \times 10^{-16}\, \sqrt{10^{12}\Delta a/a}\,{\rm yr}^{-1}$. [The most natural sign here being $+$, i.e. $b_F > 0$, which corresponds to *smaller* e^2 in the past, just as suggested by the claim [Webb *et al.*, 2001].] Such a time variation might be observable (if $\Delta a/a$ is not very much below its present upper bound $\sim 10^{-12}$) through the comparison of high-accuracy cold-atom clocks and/or via improved measurements of astronomical spectra.

More theoretical work is needed to justify the basic assumption (3.3) of our scenario. In particular, it is crucial to investigate whether it is natural to expect that the sign of the crucial coefficient b_λ in equation (3.15) be indeed *positive*. [Recall that the general mechanism of [Damour and Polyakov, 1994] is an attraction towards "Least Couplings" while equation (3.3) with $\mathcal{O}(e^{-\phi}) > 0$ leads to largest couplings at infinity.] Note in this respect that the sign of the other b_i's is not important as, once inflation has pushed $e^{c\varphi}$ to very large values $e^{c\varphi_{\rm end}}$, the subsequent cosmological evolutions tend to be ineffective in further displacing φ.

Chapter 4

Runaway dilaton as quintessence

4.1 The accelerating Universe and the dilaton

According to recent astrophysical observations, our Universe, at least since a red shift $z \sim 1$, appears to have undergone a phase of accelerated expansion[1]. This result can be combined with the recent estimates of the average mass density of the Universe [Bachall *et al.*, 1999], $\Omega_m \simeq 0.3 - 0.4$ (in critical units), and with recent measurements of the CMB anisotropy peaks [de Bernardis *et al.*, 2000], pointing at a nearly critical total energy density, $\Omega_T \simeq 1$. One is then led to the conclusion that the present cosmological evolution, when described in terms of an effective fluid entering Einstein's equations, should be (marginally) dominated by a "dark energy" component ρ_x characterized by a (sufficiently) negative effective pressure, $p_x < -\rho_x/3$.

The simplest candidate for such a missing energy is a positive cosmological constant Λ, of order H_0^2, H_0 being the Hubble parameter. Such an identification, however, unavoidably raises a series of difficult questions. In particular: a) Why is Λ so small in particle physics units? Explaining a finite but very small value for Λ may turn out to be even harder than finding a reason why it is exactly zero. This is the so-called fine-tuning problem for Λ, see for instance [Weinberg, 1989]; and b) Why is $\Lambda \sim \rho_{m0}$, where ρ_{m0} is the *present* value (in Planck units) of the (dark) matter energy density? One may note in fact that, during cosmic evolution, the cosmological constant, as well as any negative pressure component, tends to overcome very rapidely the other components. This is the so-called "cosmic coincidence" problem [Steinhardt, 1997].

At present, the most promising scenarios for solving (at least part of) the above problems introduce a single scalar field, dubbed "quintessence" [Turner and White, 1997; Caldwell, Dave and Steinard, 1998; Friemann and Vaga,

[1] See for instance [A. Riess *et al*, 1998; Perlmutter *et al.*, 1999]. The observation of a even farther supernova by [Riess *et al.*, 2001] seemed at first to imply [Turner and Riess, 2001] the acceleration period to have started not earlier than $z \sim 1$. More recently, [Amendola *et al.*, 2002] have shown that data may be consistent with an acceleration period started as early as $z \sim 5$.

1998] whose potential goes to zero asymptotically (leaving therefore just the usual puzzle of why the "true" cosmological constant vanishes). The scalar field slowly rolls down such a potential reaching infinity (and zero energy) only after an infinite (or very long) time. While doing so, quintessence produces an effective, time-dependent, cosmic energy density ρ_x accompanied by a sufficiently negative pressure, i.e. a sort of effective cosmological constant. By making $\Lambda_{eff} \sim H^2$ time-dependent, this can naturally explain the smallness of the *present* effective vacuum energy density. However, if, as in General Relativity, dust energy and an effective cosmological constant have so different time dependence, it can hardly explain why $\Lambda \sim \rho_{m0}$ i.e. the "cosmic coincidence problem" shows up. For a recent review of the relative merits of a cosmological constant and quintessence, see [Binètruy, 2000].

As far as identifying quintessence is concerned, the inflaton itself could be a candidate as recently proposed by [Peebles and Vilenkin, 1999] and by [Peloso and Rosati, 1999]. But also more exotic possibilities have been considered, in particular some motivated by the wish to solve the above-mentioned cosmic coincidence problem (see, for instance, [Hebecker and Wetterich, 2000], [Amendola and Tocchini-Valentini, 2001] and References [1]). In any case, as is the case for the inflaton, the quintessence field does not have, as yet, an obvious place in any fundamental theory of elementary particles. One should also mention, at this point, that, if quintessence may help with the problems typical of the cosmological constant interpretation, it is likely to create a new one of its own: in order to play its role, the quintessence field must be extremely light and can thus mediate a new long–range (of order H_0^{-1}) force, which is strongly constrained observationally. This is an important constraint to be imposed on any specific scalar field model of quintessence, either minimally or non-minimally (see References [2]) coupled to gravity.

At first sight, the search for a quintessence candidate in particle physics looks easier than the one for an inflaton. For instance, fundamental or effective scalar fields with potentials running to zero at infinity are ubiquitous in supersymmetric field theories and/or in string/M-theory. They are usually referred to as moduli fields since, in perturbation theory, they parametrize the space of inequivalent vacua and correspond to exactly flat directions (equivalently, to exactly massless fields). Non-perturbative effects (e.g. gauge-theory instantons) are expected to lift these flat directions, just preserving those that correspond to small or vanishing coupling. Examples are the run-away vacua of supersymmetric gauge theories (see, for instance, [Masiero, Pietroni and Rosati, 2000] for quintessence models based on the latter possibility), or the dilaton modulus ϕ in the limit $\phi \to -\infty$.

However, if we were to take one of these moduli as quintessence, we would immediately run into the problem that the acceleration of the Universe should be accompanied by a drift of interactions towards triviality. For Newton's constant, and even more so for the fine-structure constant, this kind of time variations is very strongly constrained [Uzan, 2002]. Furthermore, typical couplings of moduli fields to ordinary matter are of gravitational order, and this creates the problems described in Chapter 2 of new, unwanted long-range forces.

In this respect, the situation can be drastically improved by considering the case "$e^{\langle\Phi\rangle} > 1$, $e^{\langle\phi\rangle} < 1$" considered in Section 2.1.1: the M-theory limit where the 10-dimensional coupling $e^{\langle\Phi\rangle}$ is non-perturbative, while still keeping the four-dimensional effective couplings perturbative [Witten, 1996]. Even then, the moduli would presumably freeze out in a typical (and cosmologically tiny) particle-physics time, and therefore cannot implement the conventional, slow-roll quintessential scenario. In spite of these difficulties, unconventional models of quintessence based on the stabilization of the dilaton in the perturbative regime are not completely excluded, as recently discussed by one of us [Gasperini, 2001].

There is, however, another possibility for making the dilaton a candidate for quintessence and it is right the *strong* (4-dimensional) *coupling* regime of Section 2.3. As we have already mentioned, the region of large negative ϕ corresponds to the trivial vacuum. The idea that the Universe may have started, long before the big bang, in this region is actually the basis of the so-called pre-big bang scenario in string cosmology (for recent reviews see the references collected in [3]). Here we propose instead the possibility that the dilaton may act as quintessence at very *late* times (such as today), not by evolving towards $-\infty$ and triviality, but by going towards $+\infty$ and *strong coupling.*

In this picture there is naturally an asymptotic decoupling mechanism of ordinary matter to the dilaton, whose effective mass goes to zero at late times. The problem remains, of course, of explaining why the cosmological constant vanishes in superstring/M-theory, not only at zero coupling where supersymmetry protects it, but also at infinite (bare) coupling. Possibly, some new, stringy symmetry can explain this. It is simply assumed to be the case in this thesis.

As the dilaton is non-universally coupled to different types of matter fields, its coupling to ordinary matter can be asymptotically tiny [as in our basic assumptions (2.31)] and much stronger to typical dark matter candidates, such as the axion as first suggested by [Damour, Gibbons and Gundlach, 1990]. In that case, the interplay of the dark-matter dilatonic charge and of the dilaton potential leads to an accelerated expansion in which the relative fraction of dark energy and dark matter remains fixed (and of order 1), thus offering a possible explanation of the cosmic coincidence, as we will illustrate through explicit examples.

The chapter is a review of the work [Gasperini, Piazza and Veneziano, 2002] and is organized as follows. In Section 4.2 we present the effective string cosmology equations, in the small curvature –but arbitrary coupling– regime, with generic matter sources non-minimally coupled to the dilaton. In Section 4.3.1 we discuss analytically a possibile late-time attractor characterized by a constant positive acceleration and a fixed ratio of dark matter and dark energy. In Section 4.3.2 we provide a semi-quantitative description of the previous phase, during which the dilaton potential can be neglected. This phase is characterized by a "focusing" of the energy densities of the various components of the cosmological fluid (which occurs before the epoch of matter–radiation equilibrium), and by a subsequent "dragging" regime in which the dilaton energy density tends to follow that of non-relativistic (dark) matter. We also discuss here the main phenomenological constraints that have to be imposed on the scenario. In Section 4.4 we consider a typical example of string cosmology model including radiation, baryonic and cold dark matter, and we present the results of explicit numerical integrations. Our conclusions are summarized in Section 4.5.

4.2 Cosmological equations in the String and Einstein frames

Our starting point is the string-frame, low-energy, gravi-dilaton effective action [Green, Schwarz and Witten, 1987], to lowest order in the α' expansion, but including dilaton-dependent loop (and non-perturbative) corrections, encoded in a few "form factors" $\psi(\phi)$, $Z(\phi)$, $\alpha(\phi)$, ..., and in an effective dilaton potential $V(\phi)$ (see also [Damour and Nordtvedt, 1993, Damour and Polyakov, 1994]). In formulae:

$$\begin{aligned} S &= -\frac{M_s^2}{2}\int d^4x\sqrt{-\widetilde{g}}\,\left[e^{-\psi(\phi)}\widetilde{R} + Z(\phi)\left(\widetilde{\nabla}\phi\right)^2 + \frac{2}{M_s^2}V(\phi)\right] \\ &- \frac{1}{16\pi}\int d^4x\frac{\sqrt{-\widetilde{g}}}{\alpha(\phi)}F_{\mu\nu}^2 + \Gamma_m(\phi,\widetilde{g},\text{matter}) \end{aligned} \tag{4.1}$$

Note the slight change of notations and conventions respect to the preceding chapters. Here we follow those of [Gasperini, Piazza and Veneziano, 2002] also because they are the same as those used by [Amendola *et al.*, 2002] who calculated the perturbations in this scenario and find important constraints on the parameters. Here the metric signature is $(+,-,-,-)$, $R_{\mu\nu\alpha}{}^{\beta} = \partial_\mu\Gamma_{\nu\alpha}{}^{\beta} - \ldots$ and $R_{\mu\nu} = R_{\alpha\mu\nu}{}^{\alpha}$. We give in the following a "vocabulary" to pass from the symbols above and those that follow in this chapter to those of Chapter 3:

Chapter 3		Chapter 4
B_g	$=$	$e^{-\psi}$
$2B'_\phi + B_\phi$	$=$	$-Z(\phi)$
$d\varphi$	$=$	$k(\phi)d\phi/2 \;=\; d\hat{\phi}/(M_P\sqrt{2})$
$\alpha_{\text{dark matter}}$	$=$	$q(\phi)/k(\phi)$

Following the basic strong coupling assumptions (2.31), we shall assume that the form factors $\psi(\phi)$, $Z(\phi)$, $\alpha(\phi)$ approach a finite, physically interesting limit as $\phi \to +\infty$ while, in the same limit, $V \to 0$.

The fields appearing in the matter action Γ_m are in general non-minimally and non-universally coupled to the dilaton (also because of the loop corrections [Taylor and Veneziano, 1988]). Their gravitational and dilatonic "charge densities", $\widetilde{T}_{\mu\nu}$ and $\widetilde{\sigma}$, are defined as follows:

$$\frac{\delta\Gamma}{\delta\widetilde{g}^{\mu\nu}} = \frac{1}{2}\sqrt{-\widetilde{g}}\,\widetilde{T}_{\mu\nu}, \qquad\qquad \frac{\delta\Gamma}{\delta\phi} = -\frac{1}{2}\sqrt{-\widetilde{g}}\,\widetilde{\sigma}, \tag{4.2}$$

and it is important to stress that, when $\widetilde{\sigma} \neq 0$, the gravi-dilaton effective theory is radically different from a typical, Jordan–Brans–Dicke type model of scalar-tensor gravity [Gasperini, 1999]. We shall give a prototype form of Γ_m in the following section, after passing to the Einstein frame.

The variation of (4.1) with respect to $\widetilde{g}_{\mu\nu}$ then gives the equations:

$$\begin{aligned} &\widetilde{G}_{\mu\nu} + \psi' \,\widetilde{\nabla}_\mu \widetilde{\nabla}_\nu \phi + \left[e^\psi Z - \psi'^{\,2} + \psi''\right] \widetilde{\nabla}_\mu \phi \widetilde{\nabla}_\nu \phi \\ &+ \frac{1}{2}\widetilde{g}_{\mu\nu} \left[\left(2\psi'^{\,2} - 2\psi'' - e^\psi Z\right)(\widetilde{\nabla}\phi)^2 - 2\psi'(\widetilde{\nabla}^2\phi) - e^\psi V(\phi)\right] = \lambda_s^2 e^\psi \widetilde{T}_{\mu\nu}, \end{aligned} \tag{4.3}$$

where $\widetilde{G}_{\mu\nu}$ is the Einstein tensor, and a prime denotes differentiation with respect to ϕ. The variation with respect to ϕ, using the trace of eq. (4.3) to eliminate $\widetilde{R}$, leads to the equation

$$\begin{aligned} &\left(3\psi'^{\,2} - 2e^\psi Z\right)(\widetilde{\nabla}^2\phi) + \left[e^\psi\left(Z\psi' - Z'\right) + \psi'\left(3\psi'' - 3\psi'^{\,2}\right)\right](\widetilde{\nabla}\phi)^2 \\ &+ e^\psi\left(2\psi' V + V'\right) + \lambda_s^2 e^\psi\left(\psi'\widetilde{T} + \widetilde{\sigma}\right) = 0 \end{aligned} \tag{4.4}$$

We shall assume an isotropic, spatially flat metric background (appropriate to the present cosmological configuration), and a perfect fluid model of source. In the cosmic-time gauge we thus set

$$\begin{aligned} \widetilde{g}_{\mu\nu} = \mathrm{diag}\left[1, -\widetilde{a}^2(\widetilde{t})\ \delta_{ij}\right], \qquad & \widetilde{T}_\mu^\nu = \mathrm{diag}\left[\widetilde{\rho}, -\widetilde{p}\ \delta_i^j\right], \\ \phi = \phi(\widetilde{t}), \qquad & \widetilde{\sigma} = \widetilde{\sigma}(\widetilde{t}), \end{aligned} \tag{4.5}$$

and one can easily check, combining the above equations, that the matter stress tensor is not covariantly conserved (even in this frame), but satisfies the equation

$$\dot{\widetilde{\rho}} + 3\widetilde{H}(\widetilde{\rho} + \widetilde{p}) = \frac{\widetilde{\sigma}}{2}\,\dot{\phi}. \tag{4.6}$$

For the purpose of this paper, and for an easier comparison with previous discussions of the quintessential scenario, it is however convenient to represent the dynamical evolution of the background in the more conventional Einstein frame, characterized by a metric $g_{\mu\nu}$ minimally coupled to the dilaton, and defined by the conformal transformation $\widetilde{g}_{\mu\nu} = c_1^2 g_{\mu\nu} e^\psi$. Here c_1^2 parametrizes the asymptotic behaviour of $\psi(\phi)$,

$$c_1^2 = \lim_{\phi \to +\infty} \exp\{-\psi(\phi)\}, \tag{4.7}$$

and thus controls the asymptotic ratio between the string and the Planck scale, $M_P^2 = c_1^2 M_s^2$. In the Einstein frame the action (4.1) becomes:

$$S = -\frac{M_P^2}{2}\int d^4x\sqrt{-g}\left[R - \frac{k(\phi)^2}{2}(\nabla\phi)^2 + \frac{2}{M_P^2}\hat{V}(\phi)\right]$$

$$- \frac{1}{16\pi}\int d^4x \frac{\sqrt{-g}}{\alpha(\phi)} F_{\mu\nu}^2 + \Gamma_m(\phi, c_1^2 g_{\mu\nu} e^{\psi}, \text{matter}) , \tag{4.8}$$

where we have defined

$$k^2(\phi) = 3\psi'^2 - 2e^{\psi} Z, \qquad \hat{V} = c_1^4 e^{2\psi} V . \tag{4.9}$$

For later use it is also convenient to define a canonical dilaton field by:

$$d\hat{\phi} = \frac{M_P}{\sqrt{2}} k(\phi) d\phi , \tag{4.10}$$

although, in solving the equations, it will be easier to work directly with the original field ϕ.

We now choose, also in the Einstein frame, the cosmic-time gauge, according to the rescaling

$$\begin{aligned} &\widetilde{a} = c_1 a e^{\psi/2}, && d\widetilde{t} = c_1 dt e^{\psi/2}, && \\ &\rho = c_1^2 e^{2\psi} \widetilde{\rho}, && p = c_1^2 e^{2\psi} \widetilde{p}, && \sigma = c_1^2 e^{2\psi} \widetilde{\sigma}. \end{aligned} \tag{4.11}$$

From the $(0,0)$ and (i,j) components of eq. (4.3) we obtain, respectively, the Einstein cosmological equations (in units such that $M_P^2 = c_1^2 M_s^2 \equiv (8\pi G)^{-1} = 2$)

$$6H^2 = \rho + \rho_\phi, \tag{4.12}$$

$$4\dot{H} + 6H^2 = -p - p_\phi, \tag{4.13}$$

while from the dilaton equation (4.4) we get

$$k^2(\phi)\left(\ddot{\phi} + 3H\dot{\phi}\right) + k(\phi)\, k'(\phi)\, \dot{\phi}^2 + \hat{V}'(\phi) + \frac{1}{2}\left[\psi'(\phi)(\rho - 3p) + \sigma\right] = 0. \tag{4.14}$$

In the above equations $H \equiv \dot{a}/a$, a dot denotes differentiation with respect to the Einstein cosmic time, and we have used the definitions:

$$\rho_\phi = \frac{1}{2}k^2(\phi)\dot{\phi}^2 + \hat{V}(\phi), \qquad p_\phi = \frac{1}{2}k^2(\phi)\dot{\phi}^2 - \hat{V}(\phi). \tag{4.15}$$

The combination of equations (4.12)–(4.14) leads finally to the coupled conservation equations for the matter and dilaton energy density, respectively:

$$\dot{\rho} + 3H(\rho + p) - \frac{1}{2}\dot{\phi}\left[\psi'(\phi)(\rho - 3p) + \sigma\right] = 0, \tag{4.16}$$

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) + \frac{1}{2}\dot{\phi}\left[\psi'(\phi)(\rho - 3p) + \sigma\right] = 0. \tag{4.17}$$

For further applications, and for a more transparent numerical integration, it is also convenient to parametrize the time evolution of all variables in terms of the logarithm of the scale factor, $\chi = \ln(a/a_i)$, where a_i corresponds to the initial scale[2], and to separate the radiation, baryonic and non-baryonic matter components of the cosmological fluid by setting

$$\begin{aligned} \rho &= \rho_r + \rho_b + \rho_d \equiv \rho_r + \rho_m\,, \\ p &= \frac{1}{3}\rho_r\,, \\ \sigma &= \sigma_r + \sigma_b + \sigma_d \equiv \sigma_r + \sigma_m. \end{aligned} \tag{4.18}$$

The dilaton equation and the Einstein equation (4.12) can then be written, respectively, as

$$2\,H^2\,k^2\,\frac{d^2\phi}{d\chi^2} + k^2\left(\frac{1}{2}\rho_m + \frac{1}{3}\rho_r + \hat{V}\right)\frac{d\phi}{d\chi} + 2H^2\,k\,k'\left(\frac{d\phi}{d\chi}\right)^2 + 2\hat{V}' + \psi'\rho_m + \sigma = 0, \tag{4.19}$$

$$H^2\left[6 - \frac{k^2}{2}\left(\frac{d\phi}{d\chi}\right)^2\right] = \rho_m + \rho_r + \hat{V}. \tag{4.20}$$

The matter evolution equation (4.16) can be split into the various components as

$$\frac{d\rho_r}{d\chi} + 4\rho_r - \frac{\sigma_r}{2}\frac{d\phi}{d\chi} = 0, \tag{4.21}$$

$$\frac{d\rho_b}{d\chi} + 3\rho_b - \frac{1}{2}\left(\psi'\rho_b + \sigma_b\right)\frac{d\phi}{d\chi} = 0. \tag{4.22}$$

$$\frac{d\rho_d}{d\chi} + 3\rho_d - \frac{1}{2}\left(\psi'\rho_d + \sigma_d\right)\frac{d\phi}{d\chi} = 0. \tag{4.23}$$

Finally, eq. (4.19) is also equivalent to the dilaton conservation equation (4.17), which becomes

$$\frac{d\rho_\phi}{d\chi} + 6\rho_\phi - 6\hat{V}(\phi) + \frac{1}{2}\left(\psi'\rho_m + \sigma\right)\frac{d\phi}{d\chi} = 0. \tag{4.24}$$

4.3 Attractors

4.3.1 Accelerated late-time attractors with constant Ω_ϕ

As a first step towards a "dilatonic" interpretation of quintessence we will now discuss the possibility that the above equations, together with a string-theory motivated potential and loop corrections, are asymptotically solved by an accelerated expansion, $\ddot{a} > 0$, with frozen ratio ρ_m/ρ_ϕ of the order of unity. This last property, in particular, is expected to solve (or at least alleviate) the cosmic coincidence problem described in Section 4.1.

[2]The relation between χ and the redshift z is $\chi = -\ln(1+z) + \ln(a_0/a_i)$, where a_0 is the present value of the scale factor.

Under the basic assumption (2.31) the form factors appearing in (4.1) have a finite limit as $\phi \to +\infty$, and assuming the validity of an asymptotic Taylor expansion, we write:

$$\begin{aligned} e^{-\psi(\phi)} &= c_1^2 + b_1 e^{-\phi} + \mathcal{O}(e^{-2\phi}) \\ Z(\phi) &= -c_2^2 + b_2 e^{-\phi} + \mathcal{O}(e^{-2\phi}) \\ \alpha(\phi)^{-1} &= \alpha_0^{-1} + b e^{-\phi} + \mathcal{O}(e^{-2\phi}), \end{aligned} \tag{4.25}$$

where c_1^2, c_2^2 are assumed to be of the same order (typically 10^2) and α_0 is to be identified with the unified gauge coupling at the GUT scale, i.e. $\alpha_0 \simeq 1/25$. Unlike the model discussed in [Boisseau *et al.*, 2000] our model thus describes, in the strong coupling limit $\phi \to +\infty$, a minimally coupled, canonical scalar field $\hat{\phi} = \sqrt{2}(c_2/c_1)\phi$, see eq. (4.10). In the opposite limit, $\phi \to -\infty$, the gravi-dilaton string effective action reduces, as usual, to an effective Brans–Dicke model with parameter $\omega = -1$. We note that it is not hard to chose $\psi(\phi)$ and $Z(\phi)$ in such a way that the kinetic term of the dilaton keeps the correct sign at all values of ϕ (see the example given in Sect. 4.4).

Similarly, the assumption that V originates from non-perturbative effects, and that $V \to 0$ as $\phi \to \infty$, allows us to write, quite generically:

$$\hat{V}(\phi) = V_0 \, e^{-\phi} + \mathcal{O}(e^{-2\phi}). \tag{4.26}$$

Since the overall normalization of the potential V_0 is non-perturbative, it should be related to the asymptotic value of the gauge coupling α_0 by an expression of the form:

$$V_0 = M_s^4 \exp\left(-\frac{4}{\beta\alpha_0}\right) = M_*^4, \tag{4.27}$$

with some model-dependent (one-loop) β-function coefficient β. For a comparison with earlier studies of an exponential potential [Ferreira and Joyce, 1998], [Amendola and Tocchini-Valentini, 2001] we also note that, when referred to the canonically normalized dilaton field $\hat{\phi}$ defined in (4.10), the Einstein frame potential (4.26) asymptotically exibits an exponential behaviour

$$\hat{V} \sim \exp\left(-\frac{\lambda\hat{\phi}}{M_P}\right), \tag{4.28}$$

with $\lambda = c_1/c_2 = \sqrt{2}/k$ at $\phi \to +\infty$.

It is important to discuss the size of the potential needed for the viability of our scenario. Since the acceleration of the Universe appears to be a relatively recent phenomenon (even, possibly, an *extremely* recent one, as recently argued in [Turner and Riess, 2001]), the potential V must enter the game very late, i.e. at an energy scale of the order of $\rho^{1/4} \sim 10^{-3}$ eV. Unless we want to play with an unnaturally large present value of ϕ, this also fixes the scale of the potential in (4.26) as $V_0 \sim (10^{-3}\text{eV})^4$. As far as we know, this is feature is common to all quintessence scenarios: the problem of an outstandingly small cosmological constant is traded for the introduction of another unnaturally small mass scale M_*.

In our context, we easily find that, in order to have a properly normalized potential, we need the constant β appearing in the exponent of (4.27) to be somewhat smaller than the coefficient β_3 of the QCD beta function (see also the discussion after eq. (4.30)), say $\beta \sim 0.6\beta_3$. Given our ignorance of the origin of the dilaton potential, this looks perfectly acceptable, a priori. However, this apparent resolution of the fine-tuning problem should not hide the fact that the potential has to be adjusted very precisely if one wants to start the acceleration of the Universe not earlier that at red-shift $z \sim \mathcal{O}(1)$, and not later than today. To the best of our knowledge there is no obvious explanation, at present, of this aspect of the coincidence problem.

Let us now come to the matter sector of the action (4.8). As a typical example of Γ_m we take:

$$\Gamma_m(\phi, g, \text{matter}) = \int d^4x\sqrt{-g}\;\bar{N}\left[i\partial\!\!\!/ + m_N(\phi)\right]N + \frac{1}{2}\int d^4x\sqrt{-g}\;\left[e^{\zeta(\phi)}(\partial_\mu D)^2 - e^{\eta(\phi)}\mu^2 D^2\right] \tag{4.29}$$

the first term representing baryonic matter, the second (scalar) cold dark matter, while the gauge term appearing explicitly in (4.8) can already represent the radiation component of the cosmic fluid.

In the spirit of the strong coupling scenario we assume that ordinary matter and radiation have nearly metric couplings to $\widetilde{g}_{\mu\nu}$, i.e. that $\sigma_b, \sigma_r \simeq 0$ as $\phi \to \infty$. The relation between dilatonic charges and the terms in the actions (4.8), (4.29) are the ones discussed in Chapter 1 and 2 :

$$\frac{\sigma_b}{\rho_b} \sim \frac{\partial}{\partial\phi}\left(\ln \Lambda_{QCD}\right), \qquad \frac{\sigma_r}{\rho_r} \sim \frac{\partial}{\partial\phi}\left(\ln \alpha\right). \tag{4.30}$$

Given that $\Lambda_{QCD} \sim M_s \exp\left(-1/\beta_3\alpha\right)$ (with β_3 the coefficient of the QCD β-function), and using (4.25) for α, it is clear that both σ_b and σ_r are exponentially suppressed at large, positive ϕ.

In the dark matter sector, on the contrary, we shall assume more generic quantum corrections. By taking for instance the action in eq. (4.29), one has for the dilatonic charge of dark matter:

$$\sigma_d \;=\; -\,\zeta'(\phi)\,e^{\zeta(\phi)}(\partial_\mu\,D)^2 + \eta'(\phi)\,e^{\eta(\phi)}\mu^2 D^2. \tag{4.31}$$

Furthermore, the equations of motion for the D field give a relation between the time-averaged quantities, $e^{\zeta(\phi)}\langle\dot{D}^2\rangle = \mu^2 e^{\eta(\phi)}\langle D^2\rangle$ (which is consistent with the interpretation of D as non-relativistic matter, $p_d = 0$, as assumed in the preceding section), and relate σ_d and ρ_d by a (generally ϕ-dependent) proportionality factor

$$\sigma_d/\rho_d \;\equiv\; q(\phi) \;=\; \eta'(\phi) - \zeta'(\phi). \tag{4.32}$$

The late-time behaviour we will discuss takes place if we assume that, in the strong coupling limit (i.e., $\phi \gg 1$), $q(\phi)$ tends to a positive constant of order unity, and that the dark matter component dominates over baryonic matter and radiation. Thus, the regime we are considering is characterized [according to eqs. (4.25), (4.26)] by

$$\begin{aligned} k^2(\phi) &= 2c_2^2/c_1^2 = 2/\lambda^2, & \sigma &= \sigma_d, \\ \rho = \rho_d, \qquad q(\phi) &= q = \mathcal{O}(1), & \sigma_d &= q\,\rho_d\,. \end{aligned} \tag{4.33}$$

It follows that the dilaton coupling to the stress tensor can be asymptotically neglected with respect to the coupling to the dilatonic charge, as $\psi' \simeq e^{-\phi}/c_1^2 \ll 1$. The dilaton and dark matter conservation equations (4.23), (4.24) and the Einstein equations (4.12), (4.13) can then be written, asymptotically, in the form

$$\dot\rho_d + 3H\rho_d - \frac{q}{2}\,\rho_m\dot\phi = 0, \qquad \dot\rho_\phi + 6H\rho_k + \frac{q}{2}\,\rho_m\dot\phi = 0, \tag{4.34}$$

$$1 = \Omega_d + \Omega_k + \Omega_V, \qquad 1 + \frac{2\dot H}{3H^2} = \Omega_V - \Omega_k, \tag{4.35}$$

where we have defined

$$\begin{aligned} \rho_d &= 6H^2\Omega_d, & \rho_\phi &= \rho_k + \rho_V, \\ \rho_k &= 6H^2\Omega_k = \dot\phi^2/\lambda^2, & \rho_V &= 6H^2\Omega_V = \hat V. \end{aligned} \tag{4.36}$$

We now look for solutions with asymptotically frozen dark-matter over dark-energy ratio, and frozen "equation of state". From the constraint (4.35) this is equivalent to the requirement that ρ_k, ρ_V and ρ_d scale in the same way, i.e.

$$\frac{d\log\rho_\phi}{d\chi} = \frac{d\log\rho_d}{d\chi}, \qquad \frac{d\log\rho_V}{d\chi} = \frac{d\log\rho_d}{d\chi}. \tag{4.37}$$

The first condition and the conservation equations give

$$\frac{d\phi}{d\chi} = \frac{6}{q}\left(\Omega_V - \Omega_k\right). \tag{4.38}$$

Expressing $d\phi/d\chi$ through $\Omega_k = (d\phi/d\chi)^2/6\lambda^2$, and inserting it in the second condition (4.37), we obtain, respectively,

$$\lambda\, q = \sqrt{\frac{6}{\Omega_k}}\left(\Omega_V - \Omega_k\right), \qquad q = 2\,\frac{\Omega_V - \Omega_k}{1 + \Omega_k - \Omega_V}, \tag{4.39}$$

where in the latter the asymptotic form of the potential (4.26) has been used. The last two equations can be solved for Ω_k and Ω_V,

$$\Omega_k = \frac{6}{\lambda^2(2+q)^2}, \qquad \Omega_V = \Omega_k + \frac{q}{q+2} \tag{4.40}$$

giving easily

$$\Omega_\phi = \frac{12 + q(q+2)\lambda^2}{(q+2)^2\lambda^2}, \qquad w_\phi = -\frac{q(q+2)\lambda^2}{12 + q(q+2)\lambda^2}, \tag{4.41}$$

where the last equation for $w_\phi = (\Omega_k - \Omega_V)/(\Omega_k + \Omega_V)$ provides the dilaton's equation of state.

The above asymptotic solution, first obained by [Wetterich, 1995], and recently studied by [Amendola, 2000, Holden and Wands, 2000], generalizes the results discussed of [Ferreira and Joyce, 1998] to the interacting dark matter case, and is very similar to the results obtained by including suitable non-minimal couplings in a Brans–Dicke context [Amendola, Tocchini-Valentini, 2001], or by including an effective bulk viscosity in the dark matter stress tensor by [Zimdahl and Pavón, 2001]. Our (4.34) corresponds indeed, in the notation of [Zimdahl and Pavón, 2001], to a dissipative pressure $\Pi = -q\rho_m(\dot{\phi}/6H)$. See also [Amendola, 2000, Holden and Wands, 2000] for a discussion of the parameter values compatible with such an asymptotic solution.

Once Ω_k and Ω_V are given, one can easily compute all the relevant kinematic properties of the asymptotic solution as a function of only two parameters, q and $\lambda = c_1/c_2$, which are in principle calculable for a given string theory model. The asymptotic value of the acceleration, in particular, is fixed by eq. (4.35) as

$$\frac{\ddot{a}}{aH^2} = 1 + \frac{\dot{H}}{H^2} = \frac{q-1}{q+2}. \tag{4.42}$$

One can also easily obtain, through a simple integration, the asymptotic evolution of the Hubble factor and of the dominant energy density,

$$H \sim a^{-3/(2+q)}, \qquad \rho \sim a^{-6/(2+q)}. \tag{4.43}$$

In order to illustrate the range of parameters possibly compatible with present phenomenology, we have plotted in the $\{\lambda\ , q\}$ plane various curves at $\Omega_\phi = \Omega_k + \Omega_V = \text{const}$, and $w_\phi = (\Omega_k - \Omega_V)/(\Omega_k + \Omega_V) = \text{const}$. (Fig. 4.1). Note that the case discussed by [Ferreira and Joyce, 1998] corresponds to staying on the λ axis. In that case, the critical value of λ below which $\Omega_d/\Omega_\phi \to 0$ is $\sqrt{3}$. The addition of q makes parameter space two-dimensional, with the point $\lambda = \sqrt{3}$ replaced by the left-most curve $\Omega_\phi = 1$. Beyond that curve, i.e. for $\lambda^2 < 6/(2+q)$ (as well as for all values of λ if $q < -2$), the ratio Ω_d/Ω_ϕ goes to zero. However, while in the case of [Ferreira and Joyce, 1998] acceleration and a finite ratio Ω_d/Ω_ϕ are incompatible, this is perfectly possible in a large region of the $\{\lambda\ , q\}$ plane.

In fact, it is possible to determine the region of our parameter space that survives the various observational constraints (Type 1a supernovae, CMB anisotropies, large-scale structure ...). The present values of Ω_ϕ and w_ϕ have to lie in the range [Wang *et al.*, 2000, Balbi *et al.*, 2001] $0.6 \lesssim \Omega_\phi \lesssim 0.7$, and $-1 \le w_\phi \lesssim -0.4$, but the two allowed intervals are not uncorrelated. Assuming that we are already in the asymptotic regime, the allowed region lies roughly between the two curves $\Omega_\phi = 0.6$ and $\Omega_\phi = 0.7$ and above $q = 2$. Other phenomenological (but somewhat more model-dependent) constraints on q and λ can be obtained from the recent measurements of the position of the third anisotropy peak in the CMB distribution [de Bernardis *et al.*, 2001], which constrains the value of Ω_ϕ today and at last scattering, as well as the time-averaged equation of state $\langle w_\phi \rangle$ [Doran, Lilley and Wetterich, 2002]. In the final part of

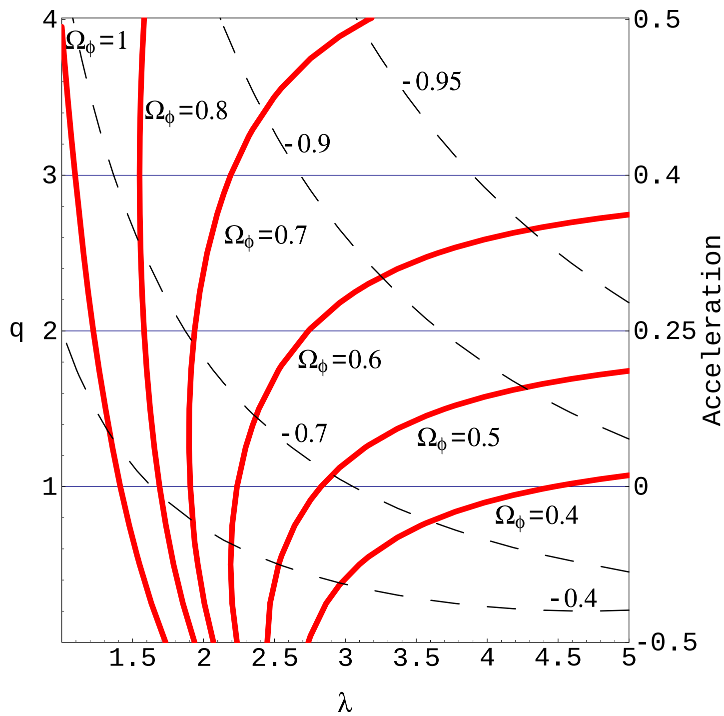

Figure 4.1: *The asymptotic configurations in the plane* $\{\lambda\ , q\}$*. The full bold curves correspond to asymptotic solutions with fixed ratios* ρ_ϕ/ρ_d *and with the following values of* Ω_ϕ*:* 1, 0.8, 0.7, 0.6, 0.5, 0.4 . *On the right vertical axis we have reported the corresponding q-dependent acceleration parameter,* $\ddot{a}a/\dot{a}^2$*. The thin dashed curves correspond to fixed asymptotic values of the dilatonic equation of state* $w_\phi = p_\phi/\rho_\phi$*, respectively* -0.4, -0.7, -0.9 *and* -0.95.

this paper we shall present a model of dark matter that seems to be compatible with all the above-mentioned constraints.

4.3.2 Before acceleration: focusing and dragging, an analytic study

Having discussed, in the previous section, the late-time accelerated expansion caused by the interplay of the dilaton potential and the dark-matter dilatonic charge, it looks appropriate to illustrate the earlier evolution, i.e. *before* the dilaton potential starts entering the game. In this section we shall provide a semi-quantitative, analytic analysis of this behaviour as it follows from the string cosmology equations (4.21)–(4.24), by imposing on the non-perturbative normalization (4.27) the constraint $V_0^{1/4} \ll H_{\rm eq}$, where $H_{\rm eq}$ is the curvature scale at the epoch of matter-radiation equality. In such a way the dilaton potential may eventually become important only at late times, in the matter-dominated era. We will show that this early evolution can be roughly divided in three epochs, providing, altogether, an intermediate attractor that nicely connects to the accelerated behaviour described in Section 4.3.1.

Let us start by considering an initial, post-big bang and post-inflationary regime of expansion driven by the standard radiation fluid, with negligible dilatonic charge, $\sigma_r = 0$. Possible non-relativistic matter, if present, is highly subdominant with respect to the other components ($\rho_m \ll \rho_\phi,\ \rho_r$) and, consequently, the dilatonic terms in eq. (4.24) can be neglected. The conservation equations can be easily integrated to give

$$\rho_r = \rho_{ri} e^{-4\chi}, \qquad \rho_\phi = \rho_{\phi i} e^{-6\chi}. \tag{4.44}$$

Therefore, the dilaton (kinetic) energy density, even if initially of the same order as ρ_r, is rapidly diluted like a^{-6}. The dilaton itself, starting from a value $\phi_i \sim \mathcal{O}(1)$ typical of the moderately-strong coupling post-big bang epoch, tends to settle down to a constant value that can be easily estimated as follows

$$\rho_k = \frac{k^2}{2} H^2 \left(\frac{d\phi}{d\chi}\right)^2 = \frac{k^2}{12}\left(\frac{d\phi}{d\chi}\right)^2 (\rho_r + \rho_\phi) = \rho_\phi. \tag{4.45}$$

For $\rho_{ri} = \rho_{\phi i}$ we get

$$\frac{d\phi}{d\chi} = \frac{\sqrt{12}}{k}\left(1 + e^{2\chi}\right)^{-1/2}, \tag{4.46}$$

which, for $k =$ const., leads to a solution with asymptotic value $\phi = \phi_1$, related to the initial value $\phi_i = \phi(0)$ by the constant shift

$$\Delta\phi = \phi_1 - \phi_i = \frac{\sqrt{12}}{k}\ln(1+\sqrt{2}) \simeq \frac{3}{k} \simeq \frac{3}{\sqrt{2}}\frac{c_1}{c_2}, \tag{4.47}$$

independently of ϕ_i and of the initial χ (the last equality holds for ϕ_1 large enough to justify the asymptotic relation $k = \sqrt{2}/\lambda$).

Such an initial regime is effective until the dilaton kinetic energy becomes of the same order as ρ_m. At that point, some oscillations are triggered by the interference term of eq. (4.24), but the dilaton energy density keeps decreasing, on the average, until it enters a "focusing" regime, during which it is diluted at a much slower rate (like a^{-2}), so as to converge, at equality, towards the larger values of ρ_m and ρ_r. Eventually, when dark non-relativistic matter becomes the dominant source ($\rho_d \gtrsim \rho_r$), the dilaton energy density tends to follow the dark matter evolution, as if it were "dragged" by it.

Before turning to a quantitative analysis of these two regimes we note that the time evolution of ρ_ϕ, in the "tracking" quintessence, is determined by the slope of the potential. In the present context, instead, the focusing and dragging effects are not due to the potential, but they are controlled by the non-minimal coupling induced by $(\psi' + q)$ (thus implementing an attractor mechanism already proposed for a class of non-minimal scalar-tensor models of quintessence [Damour and Nordtvedt, 1993], [Bartolo and Pietroni, 2000]). Thanks to the focusing effect, which seems to be typical of the string effective action (even if similar, in a sense, to the "self-adjusting" solutions of general relativity with exponential potential [Ferreira and Joyce, 1998]), the dilaton energy density at the matter–radiation equality turns out to be fixed independently from its initial value, and only slightly dependent from the initial value of the dilaton, ϕ_i. For large enough values of q, however, even the dependence upon ϕ_i tends to disappear, because the value of the dilaton itself gets focused, as will be discussed in the next section.

For a quantitative analytical study of the "focusing" and "dragging" regimes, we start from eqs. (4.21)–(4.24). Lumping together baryonic and dark matter, neglecting V, and assuming, according to (4.32), $\sigma = \sigma_m = q(\phi)\ \rho_m$, those equations can be easily recast in the form:

$$\rho_r^{-1} \frac{d\rho_r}{d\chi} + 4 = 0, \tag{4.48}$$

$$\rho_m^{-1} \frac{d\rho_m}{d\chi} + \left[3 \mp \sqrt{3}\,\epsilon\,(\rho_\phi/\rho)^{1/2}\right] = 0, \tag{4.49}$$

$$\frac{d\rho_\phi}{d\chi} + 6\rho_\phi \pm \sqrt{3}\,\epsilon\,\rho_m(\rho_\phi/\rho)^{1/2} = 0, \tag{4.50}$$

where we have introduced the important parameter:

$$\epsilon(\phi) \equiv \frac{\psi'(\phi) + q(\phi)}{k(\phi)}, \tag{4.51}$$

and the sign ambiguity comes from solving eq. (4.45) for $d\phi/d\chi$ in terms of ρ_ϕ. The focusing solution is then characterized by the relation:

$$\rho_\phi = \frac{n^2(\phi)\,\rho_m^2}{\rho}, \tag{4.52}$$

i.e. $\Omega_\phi = n^2(\phi)\,\Omega_m^2$, which holds under the assumption that both ϵ and n are slowly varying. Indeed, we can establish the connection between these two quantities by inserting the ansatz (4.52) into (4.50). This gives:

$$-6 \mp \sqrt{3}\,\frac{\epsilon}{n} = 2n^{-1}\frac{dn}{d\chi} + 2\rho_m^{-1}\frac{d\rho_m}{d\chi} - \rho^{-1}\frac{d\rho}{d\chi}, \tag{4.53}$$

where on the right–hand side the logarithmic derivative of (4.52) has been taken. By using (4.49) one finally has

$$\rho^{-1}\frac{d\rho}{d\chi} \mp \sqrt{3}\,\epsilon\left[n^{-1} + 2\,\Omega_m n\right] = 2n^{-1}\frac{dn}{d\chi} \simeq 0. \tag{4.54}$$

We can now discuss a few cases of interest. During the radiation–dominated phase, and after the kinetic energy of the dilaton is quickly red-shifted away, we can neglect the term with Ω_m in eq. (4.54), we set $d\rho/d\chi = -4\rho$, and obtain:

$$n \simeq \frac{\sqrt{3}\,\epsilon}{4}, \qquad \frac{d\phi}{d\chi} \simeq -\frac{3\,\rho_m\,\epsilon}{2\,k\,\rho}, \qquad \rho_\phi \simeq \frac{3\,\rho_m^2\,\epsilon^2}{16\,\rho}. \tag{4.55}$$

We refer to this behaviour as "focusing" since it implies that ρ_m lies, modulo a factor $(16/3)\epsilon^{-2}$, at the geometric mean between $\rho \sim \rho_r$ and ρ_ϕ. Hence, as we approach radiation-matter equality, ρ_ϕ is effectively focused towards the same common value of the other two components (see eq. (4.58) below). Note that, for a positive ϵ, this happens thanks to a *negative* $d\phi/d\chi$.

In the matter–dominated regime it is no longer safe to neglect the term in Ω_m in eq. (4.54), unless $\epsilon \ll 1$. In that case, the solution is

$$n \simeq \sqrt{3}\,\epsilon\left(-\rho^{-1}\frac{d\rho}{d\chi}\right)^{-1}, \qquad \Omega_\phi \simeq 3\,\epsilon^2\Omega_m^2\left(-\rho^{-1}\frac{d\rho}{d\chi}\right)^{-2}. \tag{4.56}$$

During matter domination, using $d\rho/d\chi = -3\rho$, one gets

$$n \simeq \frac{\epsilon}{\sqrt{3}}, \qquad \frac{d\phi}{d\chi} \simeq -\frac{2\,\epsilon}{k}, \qquad \rho_\phi \simeq \frac{\rho_m\epsilon^2}{3}. \tag{4.57}$$

In other words, the focusing regime has been turned into a dragging one: the dilaton energy is dragged along by the (dark) matter energy and keeps a (small) constant ratio to it. Incidentally, at the epoch of exact matter-radiation equality, using $d\rho/d\chi = -3.5\rho$, we easily get (still at small ϵ):

$$\frac{\rho_\phi}{\rho_{eq}} \simeq \frac{3\epsilon^2}{49}, \tag{4.58}$$

which is always smaller than 6% for $\epsilon < 1$.

In order to understand what happens at larger values of ϵ it is useful to find the reason why, for small ϵ, ρ_ϕ/ρ_m stays constant. This comes about because the corrections to the a^{-3} and a^{-6} laws for ρ_m and ρ_ϕ, due to the non-vanishing ϵ, push the two towards each other. It is easy to check that, precisely if $\rho_\phi/(\rho_m+\rho_\phi) = \epsilon^2/3$, both energies scale like $a^{-(3+\epsilon^2)}$. We note, incidentally, that the above ratio of energies nicely fits with the value given in (4.57) when $\epsilon \ll 1$. If $\epsilon < 1$, the decrease of ρ_ϕ is still slower than the a^{-4} of ρ_r, which justifies neglecting the latter. However, if $\epsilon > 1$, this is no longer the case and we have a third kind of behaviour, which can be called "total dragging". In that case, as shown by a simple analysis, all three components of ρ scale like radiation, with the following sharing of the "energy budget" (remember that we are always at $\Omega = 1$):

$$\Omega_\phi = \frac{\Omega_m}{2} = \frac{1}{3\epsilon^2}, \qquad \Omega_r = \frac{\epsilon^2 - 1}{\epsilon^2}. \tag{4.59}$$

In the next section we will see how numerical integration confirms in full detail the analytic behaviour we have discussed. We end this Section by discussing some constraints on our parameters.

As already mentioned, we assume the ordinary components of matter (radiation and baryons) to have a nearly metric coupling to $\widetilde{g}_{\mu\nu}$ (see discussion after eq. (4.29)). To be more specific, let us define the ratios between dilatonic charges and energy densities in a way similar to that used for cold dark matter in eq. (4.32), i.e.

$$q_r(\phi) \equiv \sigma_r/\rho_r, \qquad q_b(\phi) \equiv \sigma_b/\rho_b. \tag{4.60}$$

Since it is precisely the ratio $(\psi' + q_{r,b})/k$, which controls both the effective coupling of the dilaton to ordinary macroscopic matter, as well as a possible time-dependence of the fundamental constants, we shall assume that both q_b and q_r are at most of order ψ', in agreement with the discussion after eq. (4.30). We then find that there are neither appreciable violations of the equivalence principle in the context of macroscopic gravitational interactions, nor significant contributions to the time-variation of the fundamental constants, both effects being controlled by ψ'/k for $q_{r,b} \to 0$. In the strong coupling regime we have $\psi'/k \sim e^{-\phi}/(c_1 c_2)$. For a non-negative ϕ_i, and c_1^2, c_2^2 of order 10^2, there is no appreciable deviation from the standard cosmological scenario down to the epoch of matter-radiation equality, so that one easily satisfies the early-Universe constraints on dark energy, as reported for instance in [Bean, Hansen and Melchiorri, 2001].

The dilaton charge of dark matter is not restricted by the experimental tests of long–range gravitational interactions: this is the reason why we can play with it in order to produce an acceleration. Still, from the above discussion on the early phases of the universe, it is clear that high values of the dark-matter parameter ϵ may result in dangerously high values for Ω_ϕ, and thus in radical deviations from the standard cosmological scenario. Until radiation-matter equality the situation is relatively harmless: we can easily estimate the dilaton energy density at the equality and at the nucleosynthesis scale, $H_N \sim 10^{10} H_{\rm eq}$, using the fact that the dilaton, during the focusing regime, is not significantly shifted away from the value $\phi_i + \Delta\phi$, fixed by eq. (4.47). Because of the focusing behaviour we find $\Omega_\phi(\text{nucl}) \sim 10^{-10}\ \Omega_\phi(\text{eq})$, and therefore the most stringent bound comes at equality, where, thanks to (4.58), it is comfortably satisfied for $\epsilon < 1$.

During the dragging phase, however, we must certainly impose $\epsilon < 1$, otherwise, the phenomenon of "total dragging" takes place. This would represent a dramatic deviation from the standard cosmological scenario, since all the components $\rho_\phi, \rho_r, \rho_d$ (except baryonic matter) would redshift in the same way (a^{-4}) from equality until the potential starts to be felt. Even if $\epsilon < 1$, but not sufficiently small, the unusual scaling $\rho_m \propto a^{-3-\epsilon^2}$ tends to change the global temporal picture between now and the epoch of matter-radiation equality and, from eqs. (4.57), values of $\Omega_\phi \sim \epsilon^2/3$ (while in agreement with possible constraints at last scattering [Bean, Hansen and Melchiorri, 2001]) can be dangerously high. In our context, a bound $\Omega_\phi(\text{drag}) < 0.1$, i.e. $\epsilon(\text{drag}) < 0.3$, appears to be necessary in order to agree with the observed CMB spectrum and with the standard scenario of structure formation.

On the other hand, due to the smallness of $\psi' \sim e^{-\phi}/c_1^2$ in the dragging regime, an upper bound on ϵ effectively turns into a bound on the value of q/k, i.e. on the dilatonic charge of the dark matter component. Recently, the constraints on the structure formation in dilatonic quintessence model have been investigated by [Amendola *et al.*, 2002]. For a last accelerating period starting out at redshift $z \simeq 1$ they find a bound for the combination λq:

$$\lambda\ q(\phi_{\rm drag})\ <\ 0.42\,, \tag{4.61}$$

where we used the already mentioned asymptotic relation $\lambda = \sqrt{2}/k$. It is clear that a constant q cannot satisfy the above bound and, at the same time, provide the present acceleration of the Universe by means of the mechanism described in Section 4.3.1 (see also Fig. 4.1), that requires $q\lambda \gtrsim 4$.

A time– (or, better, ϕ–) dependent q, however, is allowed. For this reason we have to consider cold dark matter models like the one of eq. (4.29), whose dilatonic charge (4.32) switches on at large enough values of the dilaton. The transition to large values of ϕ is rapidly activated as the potential comes into play, $\rho_V \sim \rho_\phi$. At that point, the dilaton energy density stops decreasing and freezes at a constant value, necessarily crossing, at some later moment, the matter energy density, $\rho_\phi \sim \rho_d$. From then on, the dilaton starts rolling towards $+\infty$, triggering the effect of the dilatonic charge, which rapidly freezes the ratio ρ_ϕ/ρ_m and (for suitable values of q) leads to the accelerated asymptotic regime described by eqs. (4.41)–(4.42). Explicit numerical examples of such a behaviour will be discussed in Section 4.4.

For a realistic picture, in which the positive acceleration regime starts around the present epoch (and not much earlier) and the standard scenario of structure formation is implemented successfully, we have to require that the contribution of the dilatonic charge (as well as the effect of the dilaton potential) come into play only at a late enough epoch. The importance of this constraint was already discussed in the context of other scalar-tensor models of quintessence [Amendola and Tocchini-Valentini, 2001] where, for instance, the non-minimal coupling of the scalar field to the trace of the dark matter stress tensor was assumed to be ϕ-dependent, to interpolate between a small and a large mixing regime.

4.4 Numerical examples

Finally, after the analytic discussion of the previous section, it seems appropriate to illustrate the "run-away" dilaton scenario with some numerical example, both in order to confirm the validity of some approximations made in deriving the analytic results, and in order to see how the various regimes we discussed can be put together. To this aim, we shall numerically integrate eqs. (4.19)–(4.23), using eq. (4.20) as a constraint on the set of initial data, and assuming an explicit model for the dilatonic charges and the dilaton potential. Also, following the "induced-gravity" ideas [Sakharov, 1968], we shall specialize the loop form-factors according to eq. (4.25), using the "minimal" choice

$$e^{-\psi(\phi)} = e^{-\phi} + c_1^2, \qquad Z(\phi) = e^{-\phi} - c_2^2. \tag{4.62}$$

First of all, for a clear illustration of the "focusing" and "dragging" regimes, let us put $V = 0$, $\sigma_r = 0 = \sigma_b$, and $\sigma_d = q\rho_d$, with $q = \text{const}$. By choosing, in particular, $c_1^2 = 100$, $c_2^2 = 30$, we have integrated eqs. (4.19)–(4.23) for three different values of the charge, $q = 0$, $q = 0.01$, and $q = 0.1$, starting from the initial scale $H_i = 10^{40} H_{\rm eq}$,

$$\left(\frac{a_i}{a_{\rm eq}}\right) = \left(\frac{H_{\rm eq}}{H_i}\right)^{1/2} = \left(\frac{\rho_{mi}}{\rho_{ri}}\right) = 10^{-20}, \tag{4.63}$$

and using $\rho_{\phi i} = \rho_{ri}$, $\phi_i = -2$ as initial conditions. It should be noted that such initial conditions are generic, in the sense that different initial values of ρ_ϕ and ϕ may change the fixed value reached by ϕ during the focusing phase, but do not affect in a significant way the subsequent evolution, as will be discussed at the end of this section.

The results of this first numerical integration are illustrated in Fig. 4.2. The left panel clearly displays the initial regime of fast dilaton dilution ($\rho_\phi \sim a^{-6}$), the subsequent focusing regime ($\rho_\phi \sim a^{-2}$, see eq. (4.55)) triggered (after some oscillations) soon after ρ_ϕ falls below ρ_m, and the final dragging regime ($\rho_\phi \sim \rho_m$, see eq. (4.57)) in the epoch of matter domination (the epoch of matter-radiation equality corresponds to $\chi \simeq 46$). Note that the constant values of q have been chosen small enough to avoid the phenomenon of "total dragging", see Section 4.3.2. Note also that, in this example, ρ_m always coincides with ρ_d. In the right panel the evolution of Ω_ϕ, obtained through the numerical integration, is compared with the analytic estimates (4.55), (4.56), (4.57), for the three different values of q. In all cases, Ω_ϕ grows like a^2 during the focusing regime (in the radiation era), while the final stabilization Ω_ϕ = const, after the epoch of matter-radiation equality ($\chi \gtrsim 46$), clearly illustrates the effect of the dragging phase during which ρ_ϕ and ρ_m evolve in time with the same behaviour.

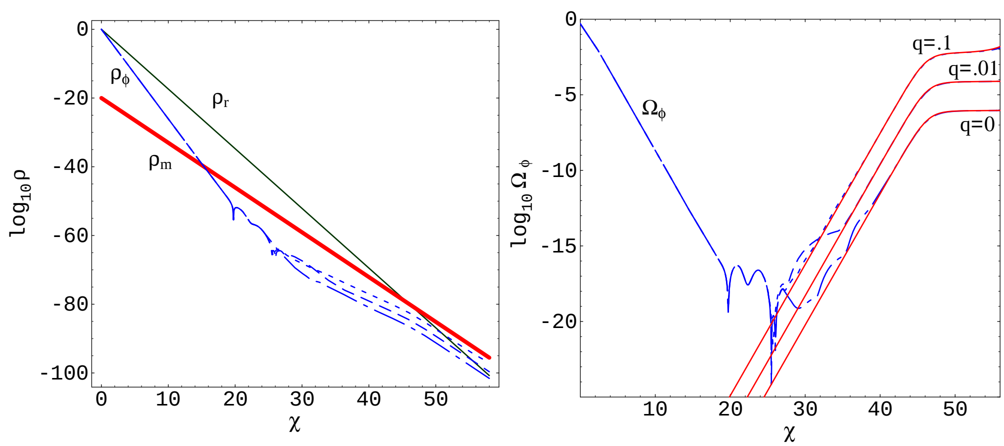

Figure 4.2: Time evolution of ρ_ϕ for $q = 0$ (dash-dotted curve), $q = 0.01$ (dashed curve) and $q = 0.1$ (dotted curve). The initial scale is $a_i = 10^{-20} a_{\text{eq}}$, and the epoch of matter-radiation equality corresponds to $\chi \simeq 46$. Left panel: the dilaton energy density is compared with the radiation (thin solid curve) and matter (bold solid curve) energy density. Right panel: the dilaton energy density (in critical units) is compared with the analytical estimates (4.55), (4.56), (4.57) for the focusing and dragging phases.

For a realistic model of quintessence, however, a constant dilatonic charge cannot drive the Universe towards an asymptotic accelerated regime and, simultaneously, satisfy all the required phenomenological constraints during the earlier epochs (as discussed in the previous sections). By keeping σ_b, $\sigma_r \simeq 0$ at large coupling (see eq. (4.30) and the discussion thereafter), we shall thus consider the explicit model of scalar dark matter (4.29), with the following simple loop form–factors

$$e^{-\zeta(\phi)} = 1 + e^{q_0\phi}/c^2, \qquad e^{\eta(\phi)} = \text{const} \tag{4.64}$$

(note that, by a field redefinition, one of the two loop factors can always be taken to be trivial: what really matters is the ratio e^{ζ}/e^{η}). Using (4.32) we immediately get

$$q(\phi) = \frac{\sigma_d}{\rho_d} = q_0 \frac{e^{q_0\phi}}{c^2 + e^{q_0\phi}} , \tag{4.65}$$

which is exponentially suppressed in the perturbative regime, and approaches $q = q_0$ at large coupling (for $q_0 > 1$ it is thus compatible with an asymptotically accelerated cosmological configuration, see Fig. 4.1). For our numerical example we shall choose $q_0 = 2.5$ and $c^2 = 150$, but the behaviour of the solution is rather stable, at late times, against large variations of the latter parameter (see the discussion at the end of this section).

In addition, we have to specify the form of the dilaton potential. In agreement with its non-perturbative origin, and with the assumtion of exponential suppression at strong coupling (see Section 4.3.2), the simplest choice is a difference of terms of the type $e^{-\beta/\alpha(\phi)}$. We shall thus consider the bell-like potential (in units $M_P^2 = 2$)

$$V(\phi) = m_V^2 \left[\exp\left(-e^{-\phi}/\beta_1\right) - \exp\left(-e^{-\phi}/\beta_2\right)\right], \qquad 0 < \beta_2 < \beta_1, \tag{4.66}$$

which leads, asymptotically, to the large-ϕ behaviour of eq. (4.26). The mass scale m_V, related to the mass M_* of eq. (4.27), will be fixed at $m_V = 10^{-3} H_{\rm eq}$, together with $\beta_1 = 10, \beta_2 = 5$, for a realistic scenario that starts accelerating at a phenomenologically acceptable epoch.

With all the parameters fixed, we have numerically integrated the evolution equations (4.19)–(4.23), for our model of charge (4.65) and potential (4.66), using the same initial conditions as in the previous example, but separating the dark and baryonic components inside ρ_m. In particular, we have set, initially, $\rho_{di} = 10^{-20}\rho_{ri}$, $\rho_{bi} = 7 \times 10^{-21}\rho_{ri}$.

The resulting late-time evolution of the various energy densities is shown in the left panel of Fig. 4.3. Dark matter and baryonic energy densities evolve in the same way, until the potential comes into play, starting at a scale around $\chi \simeq 49$. The potential first tends to stabilize ρ_ϕ to a constant but then (thanks to the contribution of q) the system eventually evolves towards a final regime in which ρ_ϕ and ρ_d are closely tied up, and their asymptotic evolution departs from the trajectory of the standard, decelerated scenario (in particular, they both scale, asymptotically as $a^{-6/(2+q_0)}$, see eq. (4.43). It is amusing to conjecture that the different time-dependence of ρ_b and r_d could be responsible for the present small ratio ρ_b/ρ_d.

In the right panel we have plotted the time evolution of the dilatonic charge q, of the energy density Ω_ϕ, of the equation of state w_ϕ, and of the

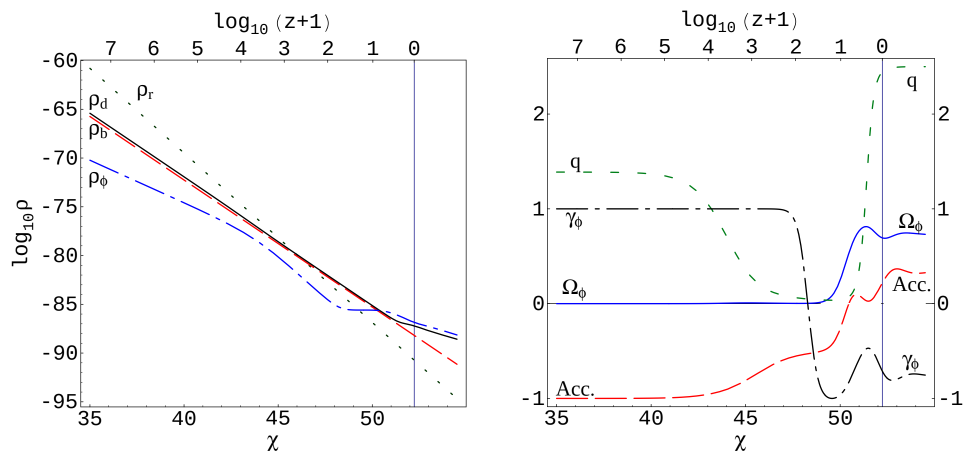

Figure 4.3: Left panel: Late-time evolution of the dark matter (solid curve), barionic matter (dashed curve), radiation (dotted curve) and the dilaton (dash-dotted curve) energy densities, for the string cosmology model specified by eqs. (4.65), (4.66). The upper horizontal axis gives the $\log_{10}$ of the redshift parameter. Right panel: for the same model, the late-time evolution of q (fine-dashed curve), w_ϕ (dash-dotted curve), Ω_ϕ (solid curve) and of the acceleration parameter $\ddot{a}a/\dot{a}^2$ (dashed curve).

acceleration parameter $\ddot{a}/aH^2$. When the potential energy becomes important, all the above quantities rapidly approach their asymptotic values given in eqs. (4.41)-(4.42). Note that, with our choice of parameters, we have $q_0 = 2.5$ and $\lambda = c_1/c_2 = \sqrt{10/3}$, corresponding to an asymptotic value $\Omega_\phi \simeq 0.733$, slightly exceeding the best fit value suggested by present observations [Wang *et al.*, 2000, Balbi *et al.*, 2001]. It is important to stress, however, that the asymptotic attractor may be preceded by a (short) oscillating regime, which, as illustrated in the right panel of Fig. 4.3, can easily allow for values of the cosmological parameters different from the asymptotic ones to be compatible with present observations. Note also that, when switching from the focusing to the dragging phases, the dilaton starts to move back towards decreasing values of q, as will be illustrated also by a subsequent numerical integration. This may slow down the evolution of ρ_ϕ with respect to ρ_m during the dragging, as shown for instance in the left panel of Fig. 4.3. Because of this effect, however, the dilaton can easily satisfy, during the dragging phase, the phenomenological bounds discussed in the previous sections. This does not require fine tuning, the validity if the bounds being guaranteed for a large basin of initial conditions by a convergent behaviour of the solutions during dragging.

During the focusing phase, in fact, the dilaton is practically frozen, as can be argued from eq. (4.55), and its effective constant value, as determined by eq. (4.47), depends on ϕ_i. However, if such a value is high enough, the presence of the dilatonic charge may become important, and may contribute to the focalization towards the epoch of matter-radiation equality, as already anticipated. This is illustrated in Fig. 4.4, which shows the time evolution of the

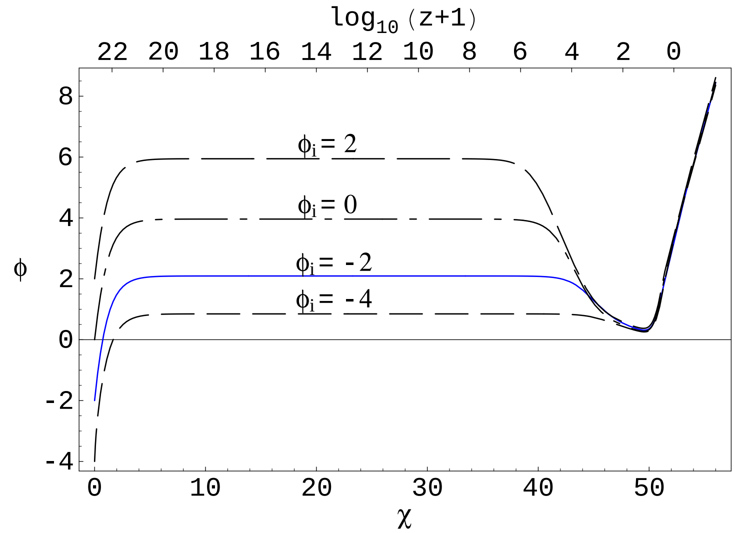

Figure 4.4: *Time evolution of the dilaton field, for different initial conditions* $\phi_i = -4, -2, 0, 2$. *All the other parameters are the same as in the example of Fig. 4.3. After the* plateau *associated with the focusing regime, and for a strong enough dilatonic charge, the solutions tend to converge to a common value of* ϕ. *The subsequent running to* $+\infty$, *driven by the potential, is thus completely independent from the initial value.*

dilaton obtained by numerically integrating the same model as in Fig. 4.3, for different initial values $\phi_i = -4, -2, 0, 2$. Although we start with different dilaton values at the *plateau* associated with the focusing regime, all the solutions tend to converge as we enter the dragging regime, so as to make the subsequent (potential-dominated) evolution *insensitive* to the initial value of the dilaton[3].

This new focusing effect, which is very different from the one of the energy densities during the radiation-dominated phase, can also be understood analytically by writing the solution of eq. (4.57) as:

$$\chi - \chi_{\rm eq} = -\int_{\phi_{\rm eq}}^{\phi} \frac{k(\bar\phi)}{2\,\epsilon(\bar\phi)}\, d\bar\phi\,. \qquad (4.67)$$

Since k is almost constant, a variation $\delta\phi_{\rm eq}$ on the initial value of ϕ changes the solution $\phi(\chi)$ by an amount $\delta\phi(\chi) = [\epsilon(\phi)/\epsilon(\phi_{\rm eq})]\delta\phi_{\rm eq}$, which rapidly decreases (with $q(\phi)$) during the dragging phase. This is why the solution has become independent of the initial value of ϕ by the time the potential becomes an important component.

For the same reason, the model is only weakly affected by variations of the parameter c in eq. (4.65), which roughly gives the transition scale between small and large dilatonic charges: $\phi_s = (2/q_0)\log c$. Indeed, because of the

[3]The *preceding* evolution, of course, is not sensitive either, since during focusing the order of magnitude of Ω_ϕ is given by Ω_m^2 as in (4.55).

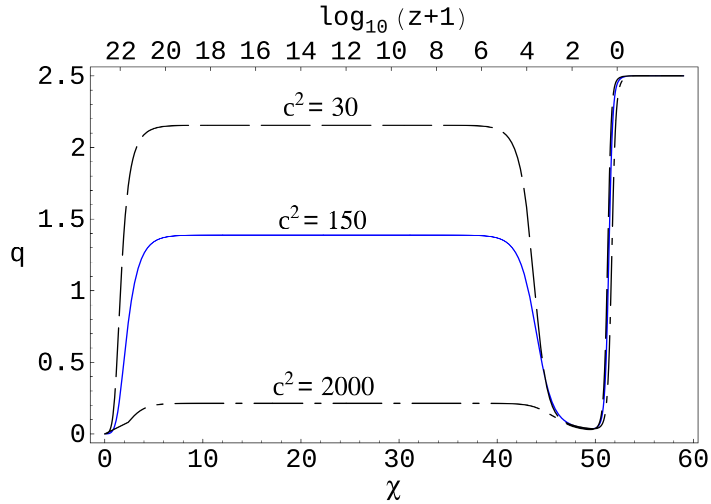

Figure 4.5: *Time evolution of $q(\phi)$, from eq. (4.65), for three different values of the parameter c. All the other parameters are the same as in the example of Fig. 4.3. During the dragging phase the value of q converges to the regime* $q \ll 1$.

above mechanism the dilaton is pushed back during the dragging phase the dilaton is pushed back, with a velocity as high as needed to reach, in any case, the safe zone $q \ll 1$. This effect is illustrated in Fig. 4.5, where we have plotted the time evolution of $q(\phi)$, for the same model as Fig. 4.3, and for three different values of c.

It should be noted, in conclusion, that the above class of models depends in crucial way on three important parameters: m_V, q_0 and the ratio $\lambda = c_1/c_2$. The first one controls the transition time between the epoch of standard cosmological evolution and the final accelerated regime (as can be easily checked, for instance, by repeating the numerical integration of Fig. 4.3 with different values of m_V). The other two parameters control the asymptotic properties of the model (acceleration, equation of state, ...), as discussed in Section 4.3.1. Future precision data, both from supernovae observations and from measurements of the CMB anisotropy, could give us a good determination of these parameters, thus providing important indirect information on the parameters of the string effective action in the strong coupling regime.

4.5 Conclusions

Let us conclude by summarizing the main points of this chapter. We have argued that a run-away dilaton can provide an interesting model of quintessence under a well-defined set of assumptions that we list hereafter:

- (*strong coupling assumption*: The limit of supertring theory, as its bare four-dimensional effective coupling goes to infinity (so called induced gravity/gauge or compositeness limit), should exist and make sense phenomenologically, i.e. should yield reasonable values for the unified gauge coupling at the GUT scale and for the ratio M_P/M_{GUT}, thanks to the large number of degrees of freedom at M_{GUT};

- In the visible-matter sector, the couplings to the dilaton, either direct or through the trace of the energy-momentum tensor (i.e. via a conformally rescaled metric), should vanish in the $\phi \to +\infty$ limit;
- In the dark matter sector, there should be a surviving coupling to the dilaton (and thus violations of the strong and/or weak equivalence principles) even in the $\phi \to +\infty$ limit;
- The dilaton potential should be non-perturbative, go to zero asymptotically, and have an absolute scale not too far from the present energy density.

Under these circumstances, it is natural for the dilaton energy in critical units, Ω_ϕ, to be: i) subdominant during radiation domination; ii) a (small) fraction of the total energy at matter-radiation equality; iii) a (small) fraction of Ω_m during the earlier epoch of matter domination; iv) a fraction of dark-matter energy since a red-shift $\mathcal{O}(1)$. This very last phase is characterized by an accelerated expansion.

In other words, this framework seems to be naturally consistent with present astrophysical observations and with known cosmological constraints. A more accurate study of the constraints of this model has recently been done by [Amendola *et al.*, 2002], expecially in relation to the cosmological perturbations and the formation of structure. Interesting enough, they find that structures keep forming even during acceleration, thanks to the stronger gravitational attraction of dark matter. As a result, they do not find any contraddiction for an accelerating phase starting as early as $z \sim 5$! If this were confirmed by further studies – such as a precise computation of the CMB anisotropy spectrum in this context – the model would gain even more interest.

Appendix A

Conformal transformations

We show how various geometric and physical quantities transform under a conformal rescaling of the metric in D space-time dimensions. The quantities relative to the conformally related frame will be denoted by a tilde. The conformal factor Ω^2 is a positive function of the spacetime coordinates, x^μ. The conformally transformed metric is then

$$\widetilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \tag{A.1}$$

and the infinitesimal line element is scaled:

$$d\widetilde{s}^2 = \Omega^2 ds^2. \tag{A.2}$$

Notice that the space-/time-like or null properties of vectors remain unaltered and that, by going to the tilde frame, lengths get a factor Ω and masses a factor Ω^{-1}. The determinant of the metric scales as

$$\sqrt{-\widetilde{g}} = \Omega^D \sqrt{-g} \tag{A.3}$$

A.0.1 Intrinsic curvature

Geometric quantities can then be defined relative to the conformally rescaled metric (A.1). The Christoffel connection for instance is

$$\widetilde{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2}\widetilde{g}^{\lambda\kappa}\left(\widetilde{g}_{\mu\kappa,\nu} + \widetilde{g}_{\nu\kappa,\mu} - \widetilde{g}_{\mu\nu,\kappa}\right) \tag{A.4}$$

$$= \Gamma^\lambda_{\mu\nu} + \frac{1}{\Omega}\left(g^\lambda_\mu \Omega_{,\nu} + g^\lambda_\nu \Omega_{,\mu} - g_{\mu\nu} g^{\lambda\kappa} \Omega_{,\kappa}\right) \tag{A.5}$$

The Riemann and Ricci tensors can similarly be defined, yielding a Ricci scalar which can be given terms of the old metric as

$$\widetilde{R} = \Omega^{-2}\left(R - 2(D-1)\Box \ln\Omega - (D-2)(D-1) g^{\mu\nu}\frac{\Omega_{,\mu}\Omega_{,\nu}}{\Omega^2}\right) \tag{A.6}$$

The d'Alembertian operator itself transforms as

$$\widetilde{\Box}\sigma = \Omega^{-2}\left(\Box\sigma + (D-2) g^{\mu\nu}\frac{\Omega_{,\mu}}{\Omega}\sigma_{,\nu}\right) \tag{A.7}$$

and we can write the original Ricci scalar in terms of the conformally transformed metric:

$$R = \Omega^2 \left(\widetilde{R} + 2(D-1)\widetilde{\Box}(\ln \Omega) - (D-2)(D-1)\widetilde{g}^{\mu\nu}\frac{\Omega_{,\mu}\Omega_{,\nu}}{\Omega^2} \right) \tag{A.8}$$

Appendix B

Induced gravitational coupling

In this appendix we calculate, with heat-kernel methods, the cosmological constant and Ricci terms induced by $N_1/2$ fermions ψ_d and N_0 ϕ_b scalars minimally coupled in D space-time dimensions. The emergence of a Ricci term and then of "gravitation" as a quantum effect is at the basis of the "induced gravity idea" [Sakharov, 1968]. Further references on the argument are [Adler, 1982], [Amati and Veneziano, 1981], [Visser, 2002].

We consider a tree-level action of the form

$$S = S_{\text{grav}} + S_s + S_f, \tag{B.1}$$

where

$$S_{\text{grav}}[g_{\mu\nu}] = \frac{1}{2\kappa_0^2}\int d^Dx\sqrt{g}\,R, \tag{B.2}$$

$$S_s[\phi_b, g_{\mu\nu}] = -\frac{1}{2}\int d^Dx\sqrt{g}\sum_b^{N_0}\left(\nabla_\mu\phi_b\nabla^\mu\phi_b + m_b^2\phi_b^2\right), \tag{B.3}$$

$$S_f[\psi_d, g_{\mu\nu}] = -\int d^Dx\sqrt{g}\sum_d^{N_{1/2}}\overline{\psi}_d\,(i\not{D} + m_d)\,\psi_d\,. \tag{B.4}$$

κ_0 is the bare gravitational coupling which, in a string theory context, are determined by the vacuum expectation value of the dilaton field.

In order to describe the effective dynamics of gravitational interactions we want to integrate out the matter fields ϕ_d and ψ_b:

$$\int \mathcal{D}[g_{\mu\nu}, \phi_b, \psi_d]\; e^{-S[g_{\mu\nu},\,\phi_b,\,\psi_d]} = \int \mathcal{D}[g_{\mu\nu}]\; e^{-S^{\text{eff}}[g_{\mu\nu}]}\,. \tag{B.5}$$

where

$$\begin{aligned} -S^{\text{eff}}_{\text{grav}} &= -S_{\text{grav}} + \sum_b^{N_0}\ln\left[\det(-\Box + m_b^2)\right]^{-\frac{1}{2}} + \sum_d^{N_{1/2}}\ln\left[\det(-\not{D}^{\,2} + m_d^2)\right]^{\frac{1}{2}} \\ &= -S_{\text{grav}} - \frac{1}{2}\sum_b^{N_0}\text{Tr}\ln(-\Box + m_b^2) + \frac{1}{2}\sum_d^{N_{1/2}}\text{Tr}\ln(-\not{D}^{\,2} + m_d^2). \end{aligned} \tag{B.6}$$

The above formal expressions can be regularized by means of the Schwinger–De Witt representation: if A is a self-adjoint operator then

$$\mathrm{Tr}\,(\ln A) \;=\; -\int_{\epsilon}^{\infty}\frac{ds}{s}\,\mathrm{Tr}(e^{-sA}), \tag{B.7}$$

as one can qualitatively check by watching at the functional dependence on the i^{th} eigenvalue λ_i ($\mathrm{Tr}\,\ln A = \sum_n \ln\lambda_n$) of the terms above :

$$-\frac{d}{d\lambda_i}\int_{\epsilon}^{\infty}\frac{ds}{s}\,\mathrm{Tr}\left(e^{-sA}\right) \;=\; \int_{\epsilon}^{\infty} ds\, e^{s\lambda_i} \quad \underset{\epsilon\to 0}{\longrightarrow} \quad \frac{1}{\lambda_i} \;=\; \frac{d}{d\lambda_i}\mathrm{Tr}\,(\ln A)\,. \tag{B.8}$$

Note that, if $A = -\Box + m_b^2$, the cut-off s has dimensions [mass^{-2}] and we can make the identification $1/s \sim \Lambda^2$.

With heat kernel techniques [Barvinsky and Vilkovisky, 1985] one can evaluate the operators in the integral of equation (B.8) in the low energy limit ($\epsilon \to 0$) for scalar and fermions respectively [Frolov, Fursaev and Zelnikov, 1997] :

$$\mathrm{Tr}\,(\exp s\,\Box) = \frac{1}{(4\pi)^{D/2}}\int d^Dx\sqrt{g}\left(s^{-D/2} + s^{1-D/2}\frac{R}{6} + \ldots\right), \tag{B.9}$$

$$\mathrm{Tr}\,(\exp s\,\not{D}^{\,2}) = \frac{1}{(4\pi)^{D/2}}\int d^Dx\sqrt{g}\left(4s^{-D/2} - s^{1-D/2}\frac{R}{3} + \ldots\right). \tag{B.10}$$

The above expansion at lowest order in s gives the induced cosmological constant term. Using (B.6) and (B.7) one has

$$\begin{aligned}\frac{\lambda}{2\kappa_0} &= \frac{1}{2(4\pi)^{D/2}}\left[\sum_b^{N_0}\int_{\epsilon}^{\infty} ds\, e^{s\,m_b^2}s^{-1-D/2} \;-4\sum_d^{N_{1/2}}\int_{\epsilon}^{\infty} ds\, e^{s\,m_d^2}s^{-1-D/2}\right]\\ &= \frac{1}{2(4\pi)^{D/2}}\left[\sum_b^{N_0} m_b^D\,\Gamma(-D/2,\, m_b^2\epsilon) - 4\sum_d^{N_{1/2}} m_d^D\,\Gamma(-D/2,\, m_d^2\epsilon)\right],\end{aligned} \tag{B.11}$$

where the incomplete gamma function

$$\Gamma(z,\sigma) \;\equiv\; \int_{\sigma}^{\infty} x^{z-1}e^{-x}dx \tag{B.12}$$

has been used. Next order of the heat kernel expansion gives the contribution to the gravitational constant:

$$\begin{aligned}\frac{1}{2\kappa^2} &= \frac{1}{2\kappa_0^2} + \frac{1}{12(4\pi)^{D/2}}\left[\sum_b^{N_0}\int_{\epsilon}^{\infty} ds\, e^{s\,m_b^2}s^{-D/2} \;+2\sum_d^{N_{1/2}}\int_{\epsilon}^{\infty} ds\, e^{s\,m_d^2}s^{-D/2}\right]\\ &= \frac{1}{2\kappa_0^2} + \frac{1}{12(4\pi)^{D/2}}\left[\sum_b^{N_0} m_b^{D-2}\,\Gamma(1-D/2,\, m_b^2\epsilon) + 2\sum_d^{N_{1/2}} m_d^{D-2}\,\Gamma(1-D/2,\, m_d^2\epsilon)\right].\end{aligned} \tag{B.13}$$

The incomplete gamma function defined in (B.12) has the following "low energy" expansion:

$$\Gamma(z, \sigma \approx 0) \;=\; -\frac{\sigma^z}{z} + \frac{\sigma^{z+1}}{z+1} - \frac{\sigma^{z+2}}{2(z+2)} + \ldots \tag{B.14}$$

by which one can finally obtain the induced gravitational terms in powers of the cut-off (remember $\epsilon \sim \Lambda^{-2}$):

$$\begin{aligned} \frac{\lambda}{2\kappa_0} &= \frac{1}{2(4\pi)^{D/2}} \left[\sum_b^{N_0} \left(\frac{2\epsilon^{-D/2}}{D} + \frac{2m_b^2 \epsilon^{1-D/2}}{2-D} \right) - 4 \sum_d^{N_{1/2}} \left(\frac{2\epsilon^{-D/2}}{D} + \frac{2m_d^2 \epsilon^{1-D/2}}{2-D} \right) \right] \\ &= \frac{1}{(4\pi)^{D/2}} \left[\left(N_0 - 4N_{1/2}\right) \frac{\Lambda^D}{D} - \left(\sum_b^{N_0} m_b^2 - 4 \sum_d^{N_{1/2}} m_d^2 \right) \frac{\Lambda^{D-2}}{D-2} \right] , \end{aligned} \tag{B.15}$$

and similarly,

$$\frac{1}{2\kappa^2} = \frac{1}{2\kappa_0^2} + \frac{1}{6(4\pi)^{D/2}} \left[\left(N_0 + 2N_{1/2}\right) \frac{\Lambda^{D-2}}{D-2} - \left(\sum_b^{N_0} m_b^2 + 2 \sum_d^{N_{1/2}} m_d^2 \right) \frac{\Lambda^{D-4}}{D-4} \right] . \tag{B.16}$$

Appendix C

The stochastic evolution of the dilaton

In this appendix we study the stochastic evolution of the dilaton φ during inflation as described by the Langevin-type equation (3.26). We restrict our attention to the region of phase space where the evolution of the inflaton χ is classical, and to a power-law potential of the form (3.14). It follows that the inflaton evolves according to the classical slow-roll equation (3.17) whose solution reads

$$\chi^2 = \chi_{\rm in}^2 - np, \tag{C.1}$$

where p, the parameter defined in (3.10), is shifted in such a way that $p_{\rm in} \equiv 0$. Equation (3.26) takes the form

$$\frac{d\varphi}{dp} = \frac{1}{2} b_\lambda \, c \, e^{-c\varphi} + \xi(p) \, , \tag{C.2}$$

where $\xi(p)$ is a gaussian stochastic variable (GSV), with a "time-dependent" r.m.s. amplitude $\widehat{H}(p)/2\pi$:

$$\langle \xi(p_1) \, \xi(p_2) \rangle \;=\; \frac{\widehat{H}^2}{(2\pi)^2} \, \delta(p_1 - p_2), \tag{C.3}$$

[the relation with the normalized random white noise term of (3.26) is $\xi(p) = \xi_2(p)\widehat{H}/2\pi$]. For any given source term $\xi(p)$, the formal solution of (C.2) reads

$$e^{c\,\varphi(p)} \;=\; e^{c\,\varphi_{\rm in}} \, e^{c\,\eta(p)} + \frac{b_\lambda c^2}{2} \int_0^p dp' e^{c\,[\eta(p) - \eta(p')]} \, , \qquad \eta(p) \equiv \int_0^p dp' \, \xi(p') \, . \tag{C.4}$$

Note that the classical solution in (3.20), $e^{c\,\varphi_{\rm cl}(p)} = e^{c\,\varphi_{\rm in}} + (b_\lambda c^2/2)\, p$, can be easily recovered in the small noise limit $\xi(p) \to 0$, $\eta(p) \to 0$.

It proves convenient to compare the true solution to the classical one by studying the statistical behaviour of the ratio $A(p) \equiv e^{c\,\varphi(p)}/e^{c\,\varphi_{\rm cl}(p)}$. As we will show below, $\langle e^{c\,\eta(p)} \rangle = \mathcal{O}(1)$. Moreover, we are also assuming $e^{c\,\varphi_{\rm in}} = \mathcal{O}(1)$

or, at least, $e^{c\varphi_{\rm in}} \ll (b_\lambda c^2/2)p$ (see Section II for details) so that the leading contribution to the first equation in (C.4) is given by the integral, and we have

$$A(p) \equiv e^{c\,\varphi(p)}/e^{c\,\varphi_{\rm cl}(p)} \simeq \frac{1}{p}\int_0^p dp' e^{c\,[\eta(p)-\eta(p')]} . \tag{C.5}$$

Since $\xi(p)$ is a GSV, also its integral $\eta(p)$ is a (centered) GSV. Moreover, if x is a GSV with $\sigma_x^2 \equiv \langle x^2\rangle - \langle x\rangle^2 = \langle x^2\rangle$, by Bloch's theorem $y = e^x$ is a new stochastic variable with $\langle y\rangle = \langle e^x\rangle = e^{\langle x^2\rangle/2}$ and $\sigma_y^2 = e^{2\langle x^2\rangle} - e^{\langle x^2\rangle}$. The average value of $A(p)$ thus reads

$$\langle A(p)\rangle \simeq \frac{1}{p}\int_0^p dp'\, e^{(c^2/2)\,\langle[\eta(p)-\eta(p')]^2\rangle} . \tag{C.6}$$

The exponent on the right hand side of the above equation can be estimated by using (C.3) and the slow-roll approximation $\widehat{H}^2 \simeq 2V(\chi,\varphi)/3 = 2\lambda(\varphi)\chi^n/3n \simeq 2\lambda_\infty\chi^n/3n$. One gets

$$\langle[\eta(p)-\eta(p')]^2\rangle = \frac{1}{(2\pi)^2}\int_{p'}^p \widehat{H}^2 dp'' \simeq \frac{n}{2(n+2)}\left[\left(\frac{\chi(p')}{\chi_{\rm in}}\right)^{n+2} - \left(\frac{\chi(p)}{\chi_{\rm in}}\right)^{n+2}\right] , \tag{C.7}$$

where $\chi_{\rm in}$ is the value at exit from self-regenerating inflation: $\widehat{H}(\chi_{\rm in})/2\pi = n/(2\chi_{\rm in})$ (see Section II for more details). Since we are interested in evaluating (C.6) at the end of inflation, $p = p_{\rm end} \simeq \chi_{\rm in}^2/n$, we can thus write

$$\langle[\eta(p)-\eta(p')]^2\rangle \simeq \frac{n}{2(n+2)}\left[1-\frac{p'}{p_{\rm end}}\right]^{(n+2)/2} . \tag{C.8}$$

When evaluated at $p' = 0$, the above formula gives $\langle\eta(p)^2\rangle = n/(2(n+2))$. Thus the normalization factor to the initial condition in (C.4) is of order one, as anticipated: $\langle e^{c\,\eta(p)}\rangle = e^{\frac{1}{2}c\langle\eta(p)^2\rangle} = \mathcal{O}(1)$. From (C.6) and (C.8) we have:

$$\begin{aligned}
\langle A(p_{\rm end})\rangle &\simeq \frac{1}{p_{\rm end}}\int_0^{p_{\rm end}} dp' \exp\left[\frac{c^2 n}{4(n+2)}\left(1-\frac{p'}{p_{\rm end}}\right)^{(n+2)/2}\right] \\
&= \int_0^1 \exp\left[\frac{c^2 n}{4(n+2)} x^{(n+2)/2}\right] dx \\
&= \exp\left(\frac{c^2 n\theta}{4(n+2)}\right) = \mathcal{O}(1),
\end{aligned} \tag{C.9}$$

with $0 < \theta < 1$.

We can estimate the dispersion of the same quantity by expanding the exponential inside the integral (C.6) in powers of $\xi(p)$:

$$A(p) \simeq 1 + \frac{c}{p}\int_0^p dp' \int_{p'}^p dp''\xi(p'') + \frac{1}{2}\frac{c^2}{p}\int_0^p dp' \left(\int_{p'}^p dp''\xi(p'')\right)^2 + \ldots . \tag{C.10}$$

At lowest order in $\xi(p)$ the variance of the above quantity calculated at $p = p_{\rm end}$ reads

$$\begin{aligned}
\sigma^2_{A(p_{\rm end})} &= \left\langle \left(\frac{c}{p} \int_0^p dp' \int_{p'}^p dp'' \xi(p'') \right)^2 \right\rangle \\
&= \frac{c^2}{p^2} \int_0^p dp' \int_0^p dp'' \int_{\max(p',p'')}^p dp''' \frac{\widehat{H}^2(p''')}{(2\pi)^2} \\
&= \frac{2c^2}{p^2} \int_0^p dp' p' \int_{p'}^p dp'' \frac{\widehat{H}^2(p'')}{(2\pi)^2} .
\end{aligned} \tag{C.11}$$

As in equation (C.7) we can use the slow-roll approximation and obtain

$$\sigma^2_{A(p_{\rm end})} = \frac{c^2\, n}{n+2} \int_0^1 x\,(1-x)^{(n+2)/2} dx = c^2 \mathcal{O}(1) .$$

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