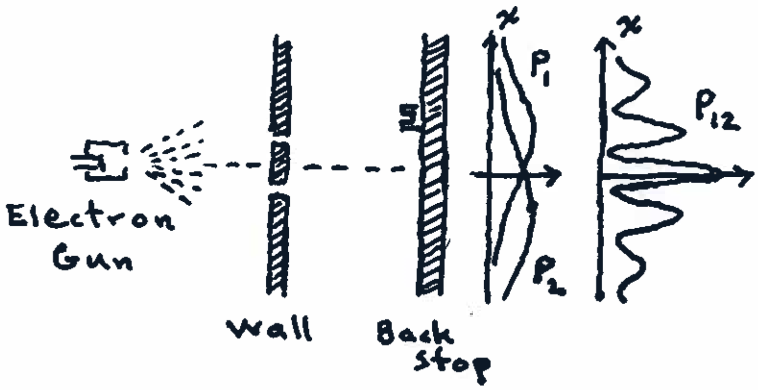
Quantum Bayesian Coherence

C. A. Fuchs
$\hat{P}I$

work with
Marcus Appleby
Ruediger Schack

Feynman 1



"The result P_{12} obtained with both holes is clearly not the sum of P_1 and P_2, the probabilities for each hole alone. In analogy with our water-wave experiment, we say: 'There is interference.'

For electrons: $P_{12} \neq P_1 + P_2$."

Instead $P_1 = |\varphi_1|^2$, $P_2 = |\varphi_2|^2$, $P_{12} = |\varphi_1 + \varphi_2|^2$.

"We shall tackle immediately the basic element of the mysterious behavior in its most strange form. We choose to examine a phenomenon which is impossible, absolutely impossible, to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the <u>only</u> mystery."

— R. P. Feynman, 1964

"Unperformed experiments have no results!"

— A. Peres, 1978

"Measurement"

Does it reveal a pre-existing, but unknown, value?

or

Does it in some sense go toward creating the very value?

EPR Criterion of Reality

"If, without in any way disturbing a system [one can gather the information required to] predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."

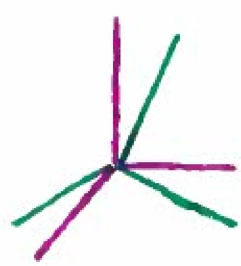
Motivated by EPR

Consider two spatially separated qutrits in a maximally entangled state:

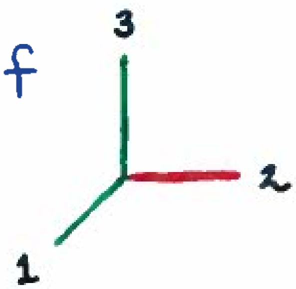
$$|EPR\rangle = \sum_{i=1}^{3} |i\rangle|i\rangle$$

Assume locality.

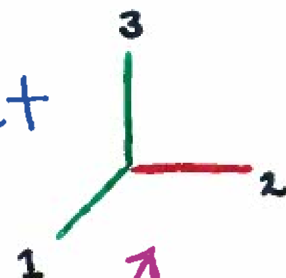
Now measure the left one any way you like. Say with A or B, two nondegenerate noncommuting observables.



So measurement is simple revelation after all?

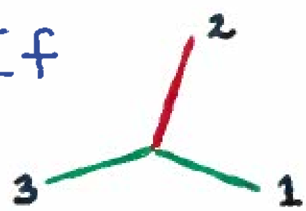
If 

here,

can predict 

there.

element of reality

If 

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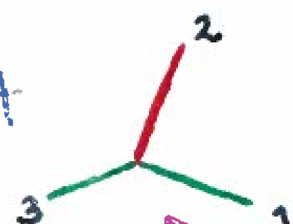
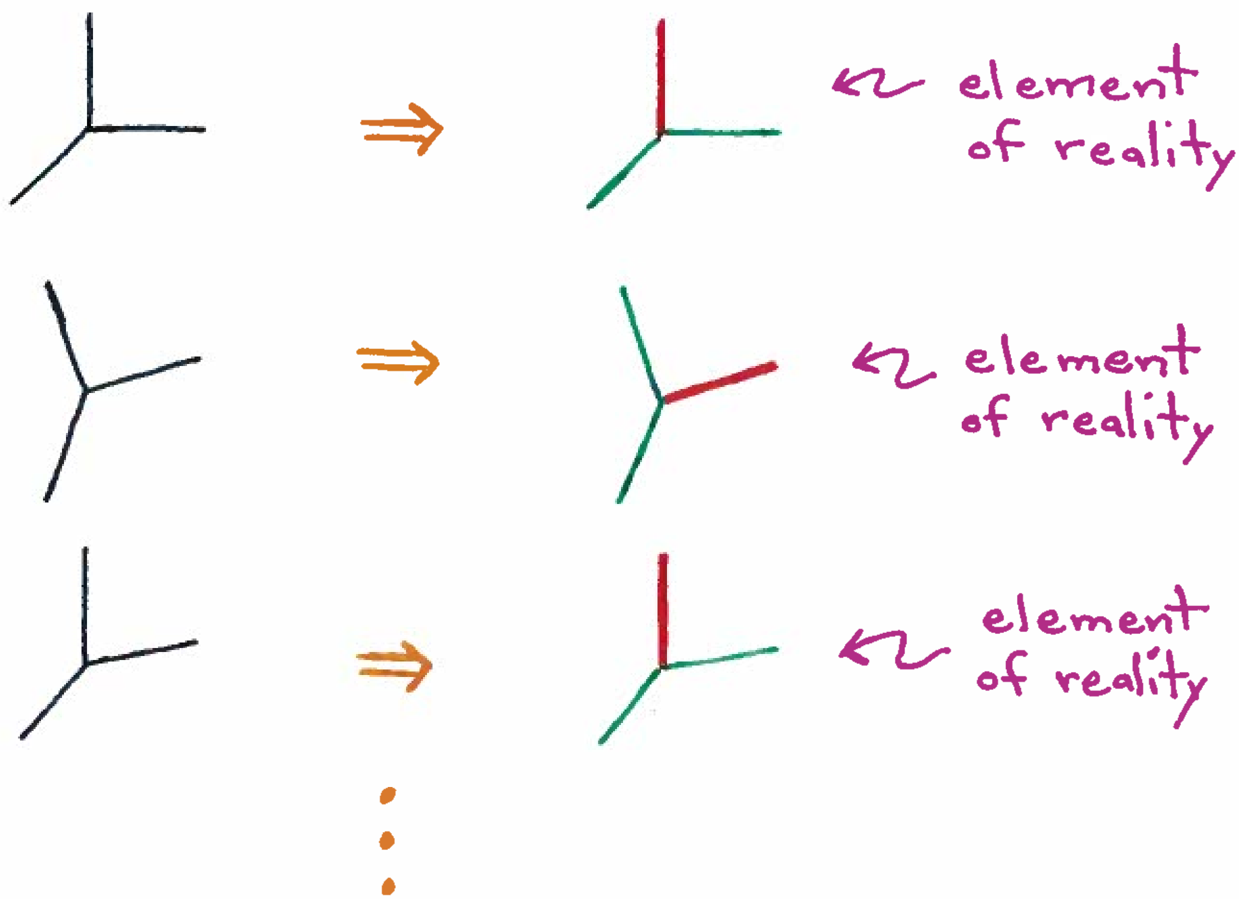
can predict 

there.

element of reality

EPR Still Implodes

But must consider many more bases than two. (~44-46)



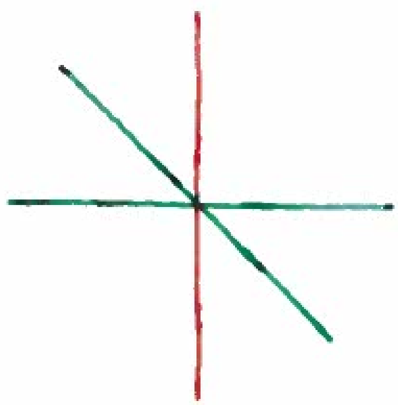
Until contradiction.

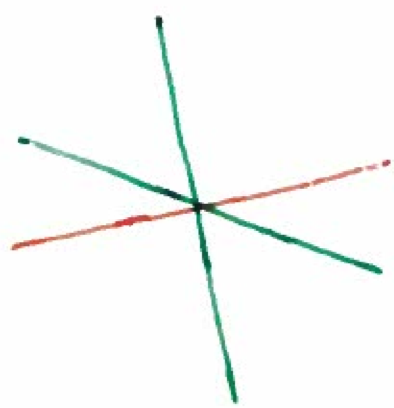
(Hint, think of Kochen-Specker.)

Kochen-Specker

Suppose they did pre-exist.

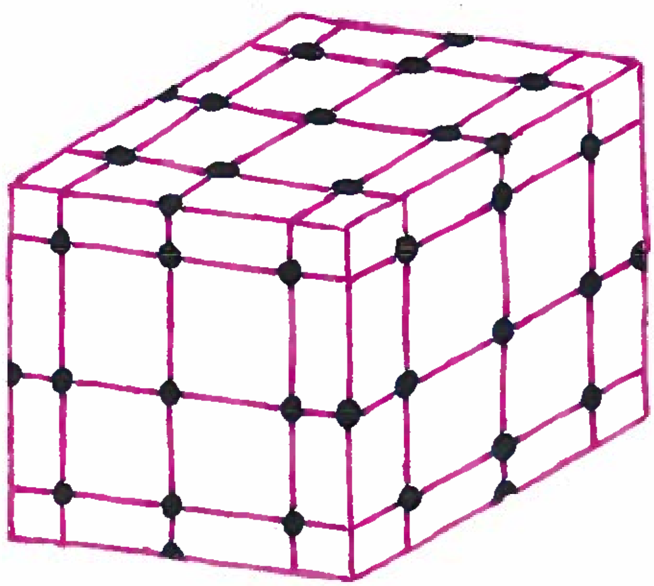
Then we should be able to color every set of orthogonal rays in $\mathbb{R}^3$ red-green-green.





Kochen - Specker

Cannot be colored:



33 rays, Peres

(when completed into full triads, consists of 40 triads made from 57 rays)

Cabello's 18-Ray Proof in $\mathcal{H}_4$

0001	0001	1-11-1	1-11-1	0010	1-1-11	11-11	11-11	111-1
0010	0100	1-1-11	1111	0100	1111	111-1	-1111	-1111
1100	1010	1100	10-10	1001	100-1	1-100	1010	1001
1-100	10-10	0011	010-1	100-1	01-10	0011	010-1	01-10

Each column represents an orthonormal basis.
So, in each column one ray will be assigned 1, and
the other three 0, at the conclusion of mmt.
Summing the values gives 9.
But each ray appears twice; preexistence of
values would then necessitate an <u>even</u> result.

9 is not even.

Contradiction!

Quantum measurements are generative:

Their outcomes do not pre-exist before the measurement interaction; they arise from the very process.

$P(h)$

$$P(h)$$

~~states of pre-existent reality~~

consequences of "measurement" interactions

Feynman 2

Is it true, or is it not true that the electron either goes through hole 1 or it goes through hole 2? The only answer that can be given is that ... there is a certain special way that we have to think in order that we do not get into inconsistencies. What we must say (to avoid making wrong predictions) is the following. If one looks at the holes or, more accurately, if one has a piece of apparatus which is capable of determining whether the electrons go through hole 1 or hole 2, then one can say that it goes either through hole 1 or hole 2. But, when one does not try to tell which way the electron goes, when there is nothing in the experiment to disturb the electrons, then one may not say that an electron goes either through hole 1 or hole 2. If one does say that, and starts to make any deductions from the statement, he will make errors in the analysis. This is the logical tightrope on which we must walk if we wish to describe nature successfully.

— R. P. Feynman, 1964

Feynman 3

From "The Concept of Probability in Quantum Mechanics," 1951:

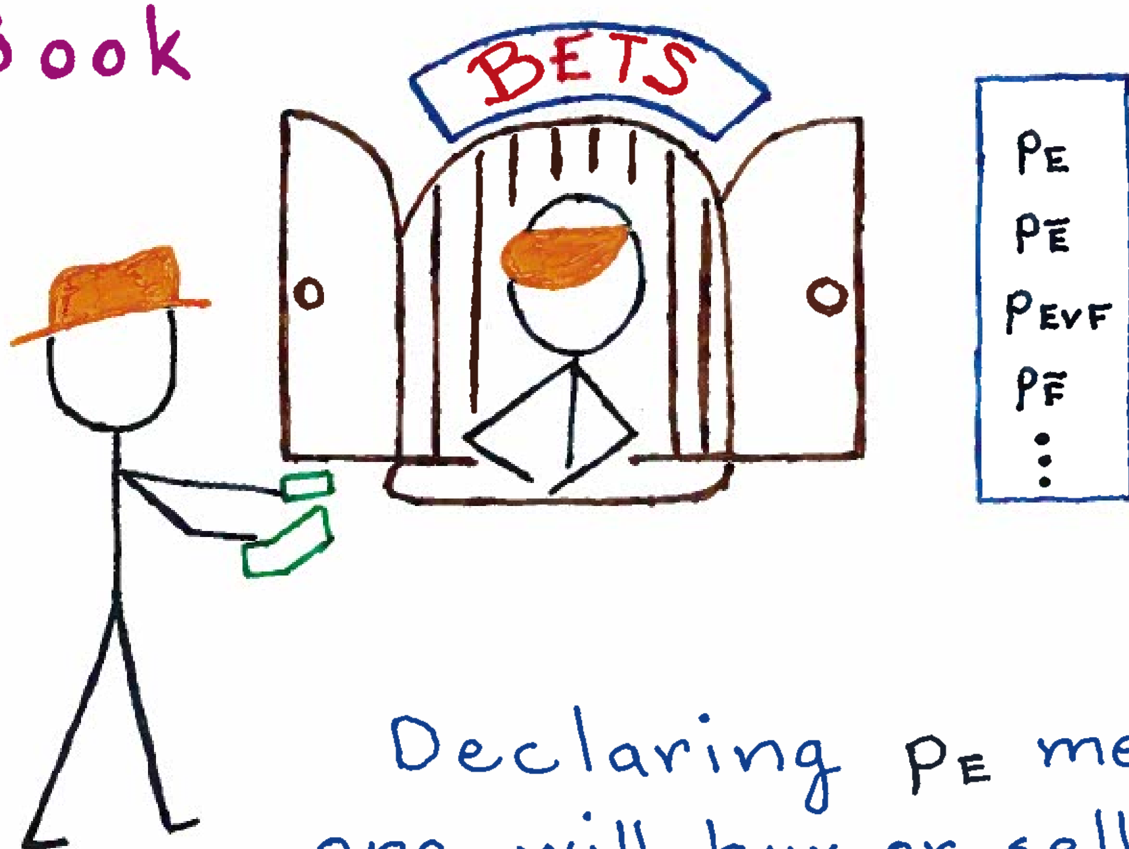
The new theory asserts that there are experiments for which the exact outcome is fundamentally unpredictable, and that in these cases one has to be satisfied with computing probabilities of various outcomes. But far more fundamental was the discovery that in nature the laws of combining probabilities were not those of the classical probability theory of Laplace.

I should say, that in spite of the implication of the title of this talk the concept of probability is not altered in quantum mechanics. When I say the probability of a certain outcome of an experiment is p, I mean the conventional thing, that is, if the experiment is repeated many times one expects that the fraction of those which give the outcome in question is roughly p. I will not be at all concerned with analyzing or defining this concept in more detail, for no departure from the concept used in classical statistics is required.

What is changed, and changed radically, is the method of calculating probabalities.

Defining Probability

Dutch Book



Declaring p_E means one will buy or sell a lottery ticket

$\boxed{\text{Worth \$1 if } E}$

for $\$p_E$.

Dutch Book

Normative Rule:

Never declare p_E, $p_{\bar{E}}$, $p_{E \vee F}$, etc. that will lead to sure loss.

Example 1:

If $p_E < 0$, bookie will sell ticket for negative money. Sure loss!

Example 2:

If $p_E > 1$, bookie will buy ticket for more than it is worth in best case. Sure loss.

Example 3:

Suppose E and F mutually exclusive.

$\boxed{\text{Worth \$1 if } E \vee F}$

$\boxed{\text{Worth \$1 if } E}$

$\boxed{\text{Worth \$1 if } F}$

buying this is equivalent to buying these two

So must have $p_{E \vee F} = p_E + p_F$.

Example 4:

$\boxed{\text{Worth } \$\tfrac{m}{n} \text{ if } E}$

Price? $\$\frac{m}{n} p_E$ of course.

Bayes Rule

Consider conditional lotteries:

If $E \wedge F$ give full price, but if $\overline{F}$ return money.

Thus:

$\boxed{\text{Worth } \$1 \text{ if } E \wedge F; \text{ Worth } \$p_{E|F} \text{ if } \overline{F}.}$ price $\$p_{E|F}$

But:

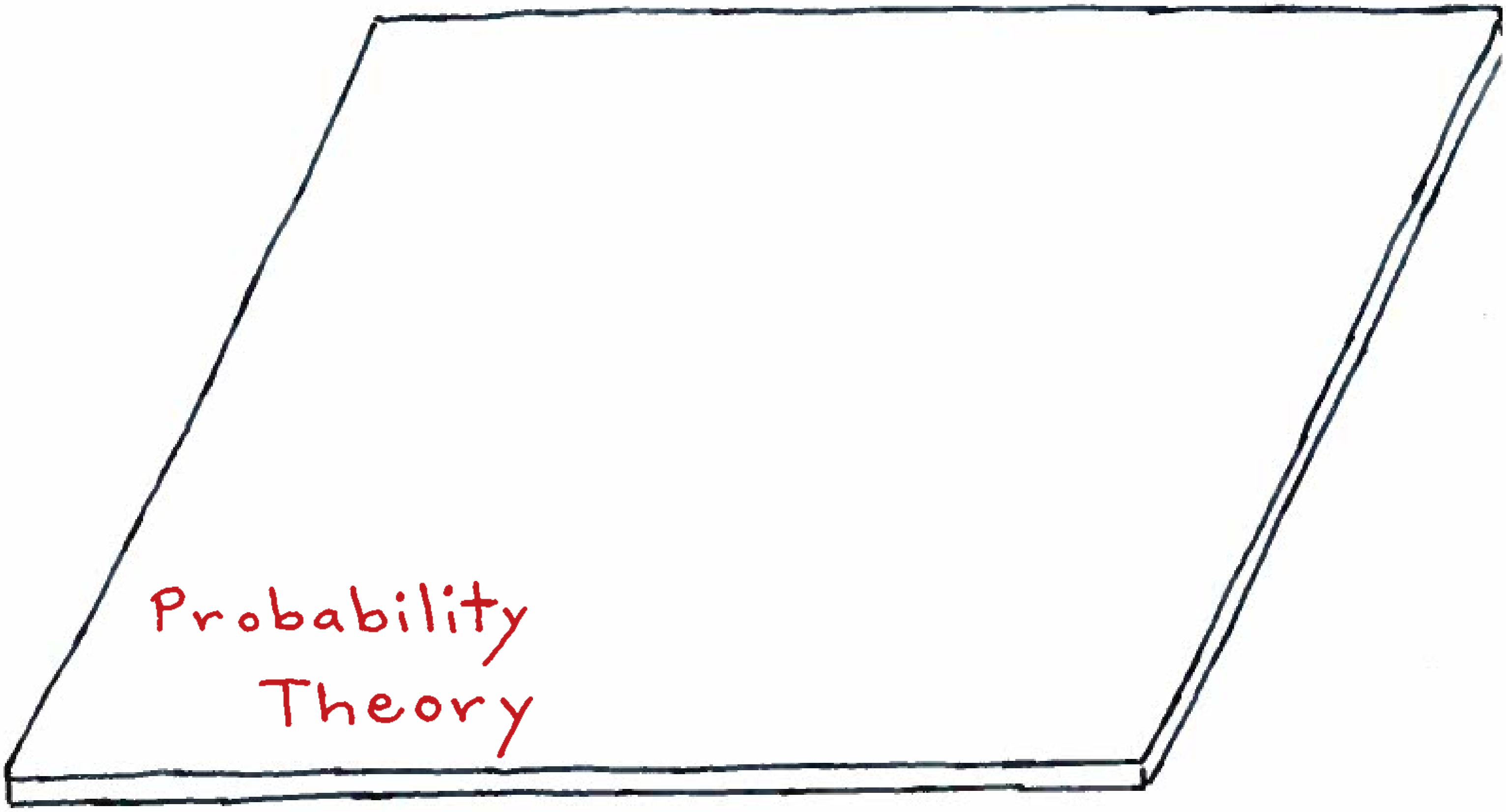
$\boxed{\text{Worth } \$1 \text{ if } E \wedge F}$ price $\$p_{E \wedge F}$

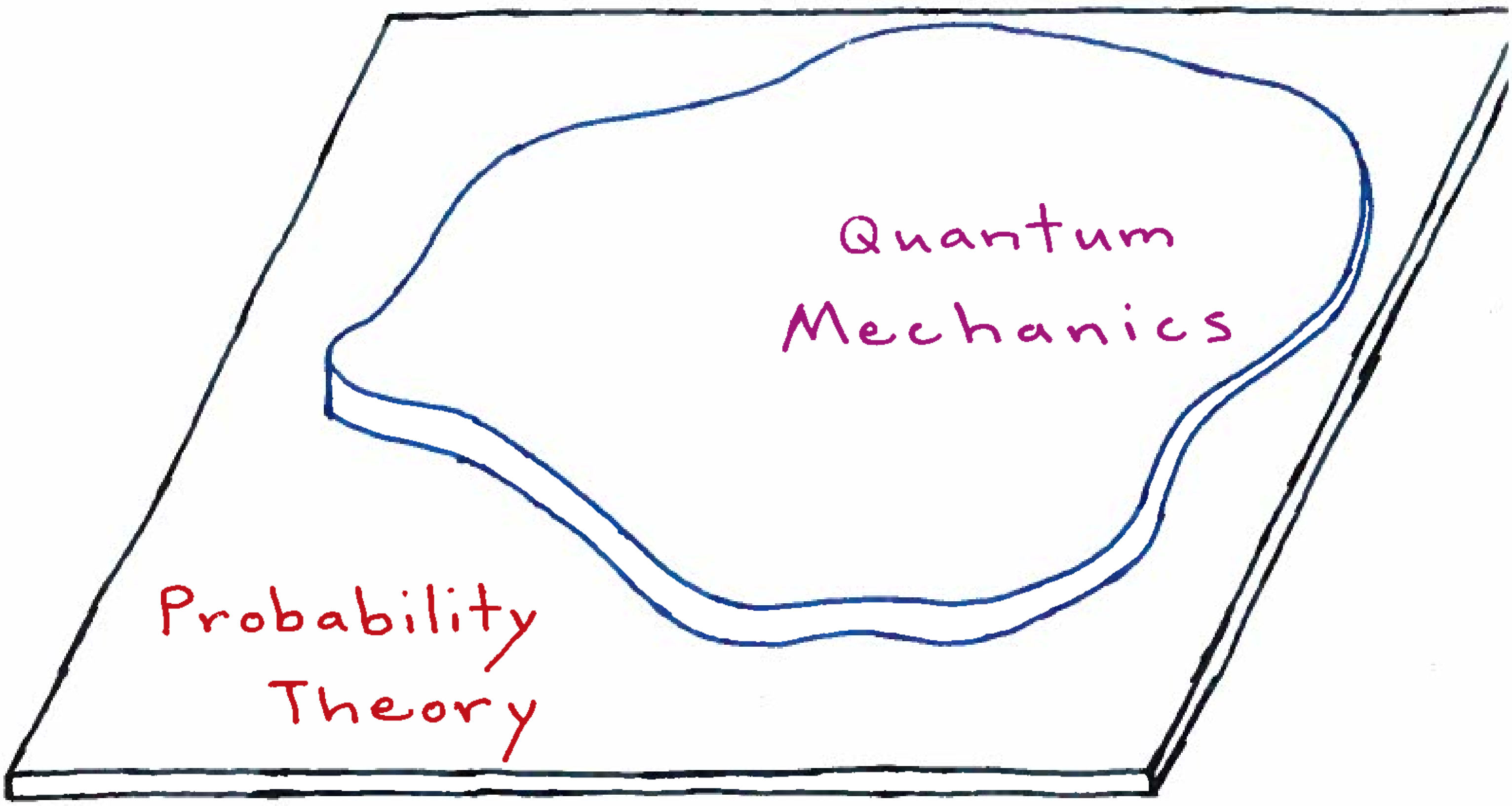
$\boxed{\text{Worth } \$p_{E|F} \text{ if } \overline{F}}$ price $\$p_{E|F}\, p_{\overline{F}}$

recall example 4

So must have:

$$p_{E|F} = p_{E \wedge F} + p_{E|F}\, p_{\overline{F}} \Rightarrow \boxed{p_{E \wedge F} = p_F\, p_{E|F}}$$


Probability
Theory


Quantum
Mechanics
Probability
Theory

Like Lewis's Principal Principle?

For David Lewis (and many others) there are Bayesian probabilities and there are "objective chances".

Principal Principle: For any agent,

$$\Pr\big(A \mid \underbrace{ch(A) = x}_{\text{chance}} \;\&\; \underbrace{E}_{\text{any "compatible" proposition}}\big) = x .$$

Similarly?: For any observer,

$$\Pr\big(j \mid ch(j) = \underbrace{|\langle j|\psi\rangle|^2}_{\text{i.e. states } |\psi\rangle \text{ give objective chances}} \;\&\; E\big) = |\langle j|\psi\rangle|^2$$

Jim Hartle 1968 *(Section IV)* Interpretation of Quantum Mechanics *(suitably modified)*

Am. J. Phys. **36**, 704–712 (1968)

A quantum state, being a summary of the observers' information about an individual physical system, changes both by dynamical laws and whenever the observer acquires new information about the system through the process of measurement. The existence of two laws for the evolution of the state vector becomes problematical only if it is believed that the state vector is an objective property of the system. If the state of a system is defined as a list of [*experimental*] propositions together with [*their probabilities of occurrence*], it is not surprising that after a measurement the state must be changed to be in accord with the new information. The "reduction of the wave packet" does take place in the consciousness of the observer, not because of any unique physical process which takes place there, but only because the state is a construct of the observer and not an objective property of the physical system.

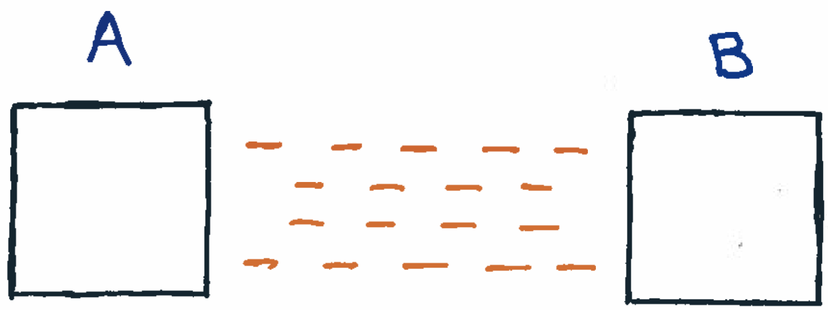
The More Pure Einstein

Granted: "The individual system (before the measurement) has no definite value of q (or p). The value of the measurement only arises in cooporation with the unique probability which is given to it in view of the ψ-function only through the act of measurement itself."

Consider spatially separated systems S_1 and S_2 initially attributed with an entangled quantum state ψ_{12}.

"Now it appears to me that one may speak of the real factual situation at S_2. ... [O]n one supposition we should, in my opinion, absolutely hold fast: the real factual situation of the system S_2 is independent of what is done with S_1. ... According to the type of measurement which I make of S_1, I get, however, a very different ψ_2 for $[S_2]$. ... For the same real situation of S_2 it is possible therefore to find, according to one's choice, different types of ψ-function.

If now [physicist B] accepts this consideration as valid, then [he] will have to give up his position that the ψ-function constitutes a complete description of a real factual situation. For in this case it would be impossible that two different types of ψ-functions could be coordinated with the identical factual situation of S_2."



$$|\Phi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$$

Let Alice measure $|\uparrow\rangle, |\downarrow\rangle$ basis.
Bob's system will be in state
$|\uparrow\rangle$ or $|\downarrow\rangle$ afterward.

Let Alice measure $|\rightarrow\rangle, |\leftarrow\rangle$ basis.
Bob's system will be in state
$|\leftarrow\rangle$ or $|\rightarrow\rangle$ afterward.

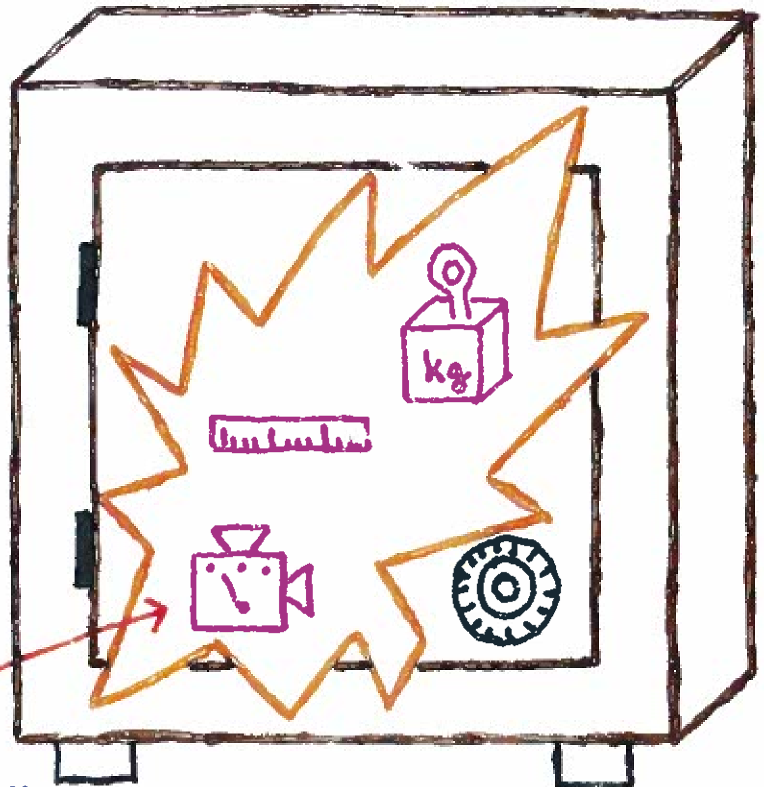
Conclusion

$|\psi\rangle$ is information.

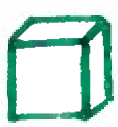
Rather, we see the Born rule as an (empirical) addition to <u>Dutch-book coherence</u>.

$$\rho \longleftrightarrow p(h)$$

Bureau of Standards

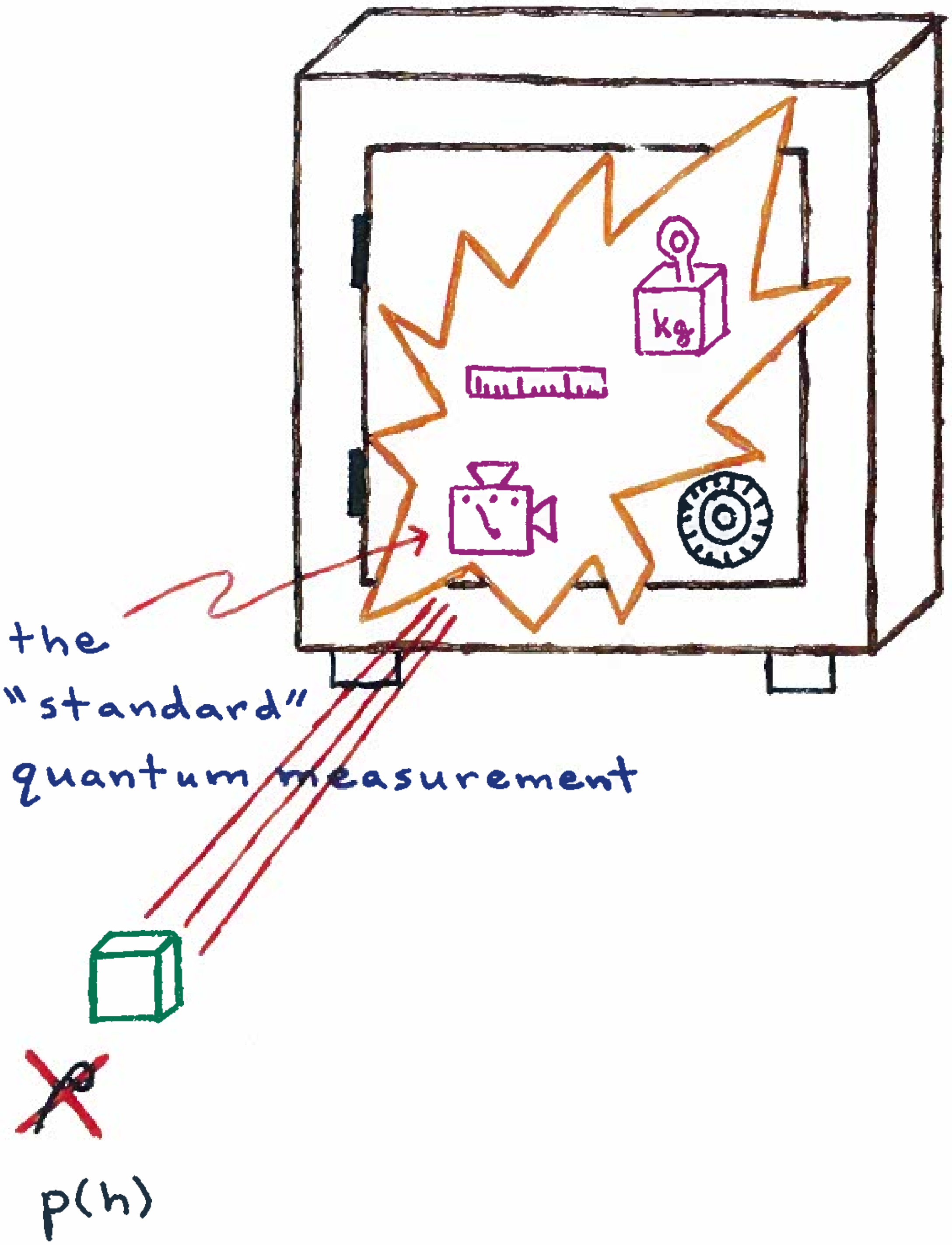


the
"standard"
quantum measurement



ρ

Bureau of Standards



Standard measurements not good enough for the bureau.

$$H = \sum_i \alpha_i \Pi_i \quad , \qquad \Pi_i = |i\rangle\langle i|$$

$$p(i) = \mathrm{tr}\,\rho\,\Pi_i = \langle i|\rho|i\rangle$$

$$\Rightarrow \begin{pmatrix} \rho_{11} & & \\ & \rho_{22} & \\ & & \ddots \end{pmatrix}$$

Standard Measurements	Generalized Measurements
$\{\Pi_i\}$	$\{E_b\}$
$\langle\psi\|\Pi_i\|\psi\rangle \geq 0, \forall \|\psi\rangle$	$\langle\psi\|E_b\|\psi\rangle \geq 0, \forall \|\psi\rangle$
$\sum_i \Pi_i = I$	$\sum_b E_b = I$
$p(i) = \mathrm{tr}\, \rho \Pi_i$	$p(b) = \mathrm{tr}\, \rho E_b$
$\Pi_i \Pi_j = \delta_{ij} \Pi_i$	—

Informational Completeness

quantum states

$\rho \in \mathcal{L}(\mathcal{H}_D)$ — D^2-dimensional vector space

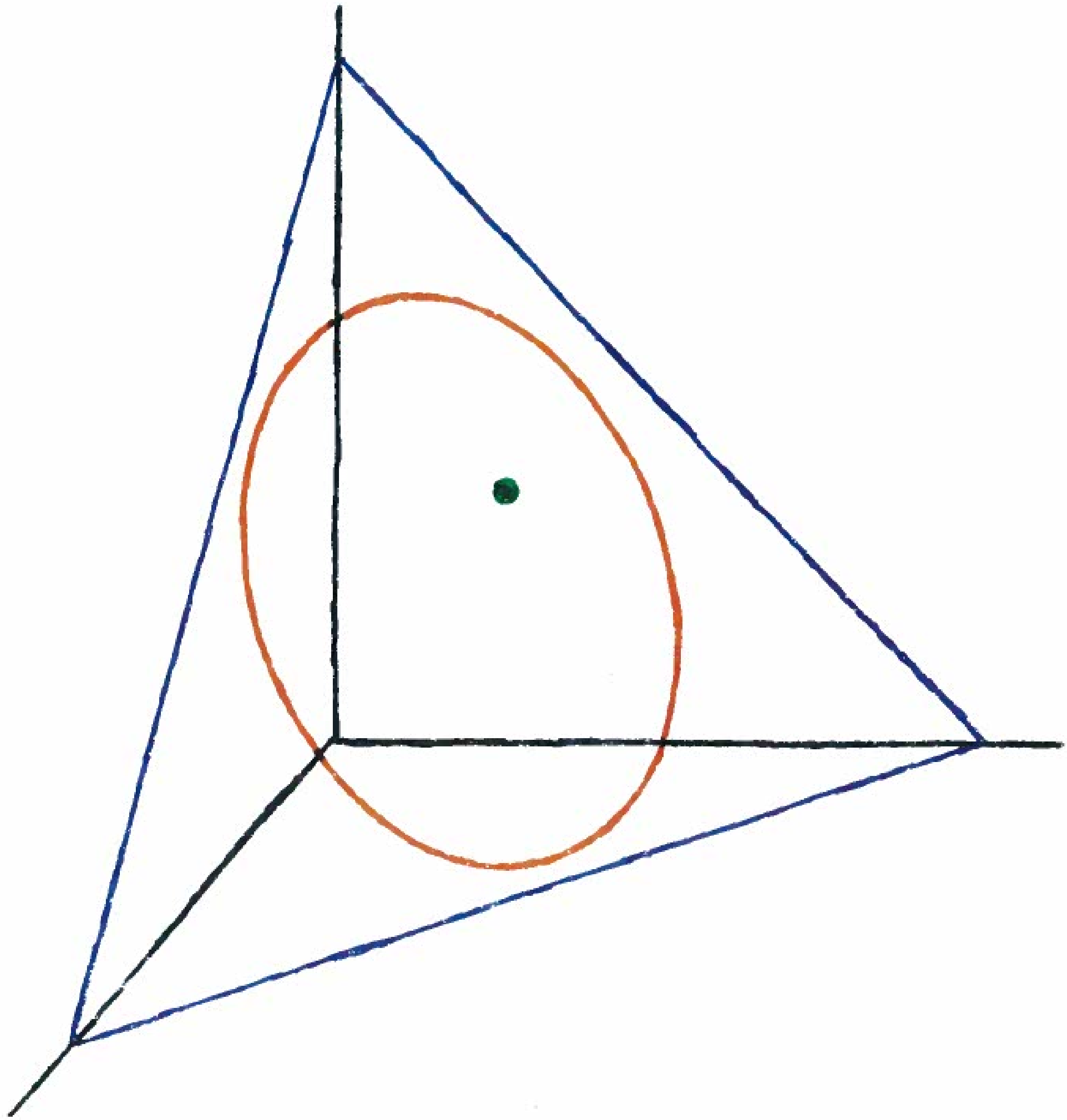
Choose POVM $\{E_h\}$, $h = 1, \ldots, D^2$, with E_h all linearly independent.
(Can be done.)

D^2 numbers $p(h) = \operatorname{tr} \rho E_h$ determine ρ.

Because $(A,B) = \operatorname{tr} A^\dagger B$ is an inner product.

projection of ρ onto E_h

Any such $\{E_h\}$ can be the standard quantum measurement.



Path Back to Density Ops

Suppose $\{E_j\}$, $j = 1, \ldots, d^2$, is ICP.

Then $p(j)$ determines ρ.

But also $\rho = \sum_j \alpha_j E_j$ for some α_j's.

Thus

$$p(j) = \mathrm{tr}\, \rho E_j = \sum_k \alpha_k \,\mathrm{tr}\, E_j E_k$$

i.e.

$$\vec{p} = M \vec{\alpha} \quad \text{where} \quad M = [\mathrm{tr}\, E_j E_k]$$

nonnegative matrix

and so

$$\boxed{\vec{\alpha} = M^{-1} \vec{p}}$$

Prettiest when $M_{jk} = a + b\,\delta_{jk}$.

SIC World (1999)

"symmetric informationally complete"

For $|\psi_i\rangle \in \mathcal{H}_d$ (d-dimensional)

find d^2 projections $\Pi_i = |\psi_i\rangle\langle\psi_i|$

such that

$$\mathrm{tr}\,\Pi_i\Pi_j = \text{constant}$$

for $i \neq j$.

Caves
Zauner

Linear Independence

Suppose m projections $\Pi_i \in \mathcal{L}(\mathcal{H}_d)$ with $\operatorname{tr} \Pi_i \Pi_j = c$ for $i \neq j$. $\quad c \neq 1$

Let α_i be numbers s.t. $\sum_i \alpha_i \Pi_i = 0$.

Then $\sum_i \alpha_i = 0$. trace of eqn above

Also

$$\sum_i \alpha_i \operatorname{tr} \Pi_i \Pi_j = 0$$

$$\Rightarrow \quad \alpha_j + c \sum_{i \neq j} \alpha_i = 0$$

$$\Rightarrow \quad (1 - c)\, \alpha_j = 0$$

$$\alpha_j = 0$$

$\Rightarrow$ The Π_i are linearly indep.

$\mathcal{H}_d = \mathbb{R}^d \Rightarrow m \leq \frac{1}{2} d(d+1)$

$\mathcal{H}_d = \mathbb{C}^d \Rightarrow m \leq d^2$

The Case $\mathcal{H}_d = \mathbb{R}^d$

d	max m	$\frac{1}{2}d(d+1)$
2	3	3
3	6	6
4	6	10
5	10	15
6	16	21
7	28	28
$\vdots$	$\vdots$	$\vdots$
15	36	120

A Very Fundamental Mmt ?

Caves, 1999
Zauner

Suppose d^2 projectors $\Pi_i = |\psi_i\rangle\langle\psi_i|$ satisfying

$$\operatorname{tr} \Pi_i \Pi_j = \frac{1}{d+1} \quad , \quad i \neq j$$

exist.

Can prove:

1) the Π_i linearly independent

2) $\sum_i \frac{1}{d} \Pi_i = I$

So good for Bureau of Standards.

Also

$$p(i) = \frac{1}{d} \operatorname{tr} \rho \Pi_i$$

$$\boxed{\rho = \sum_i \left[(d+1) p(i) - \frac{1}{d} \right] \Pi_i}$$

Evidence for Existence

Analytical Constructions

$d = 2 - 13, 15, 19$

Numerical ($\epsilon \leq 10^{-11}$)

$d = 2 - 47$

Pure States in SIC Language

Conditions

$$\rho^\dagger = \rho \quad , \quad \mathrm{tr}\,\rho^2 = \mathrm{tr}\,\rho^3 = 1$$

translate to

$$\sum_i p(i)^2 = \frac{2}{d(d+1)}$$

and

$$\sum_{jkl} c_{jkl}\, p(j)\, p(k)\, p(l) = \frac{d+7}{(d+1)^3}$$

where

$$c_{jkl} = \mathrm{Re}\,\mathrm{tr}\,\Pi_j \Pi_k \Pi_l$$

Could these be independently motivatable physical constants?

Building Hilbert Space

Given c_{ijk}, prove representation theorem!

For instance, can define a Jordan product:

$$\Pi_i \bullet \Pi_j \equiv \sum_k \alpha_{ijk} \Pi_k$$

with

$$\alpha_{ijk} = \frac{1}{d}(d+1)\left[c_{ijk} - \frac{d\delta_{ij}+1}{(d+1)^2} \right] .$$

Measure observable $\{P_j\}$.

Probability of outcome j given by

$$\boxed{p(j) = \operatorname{tr} \rho P_j}$$

"The Born Rule"

Laws of Probability

H_i – various hypotheses one might have

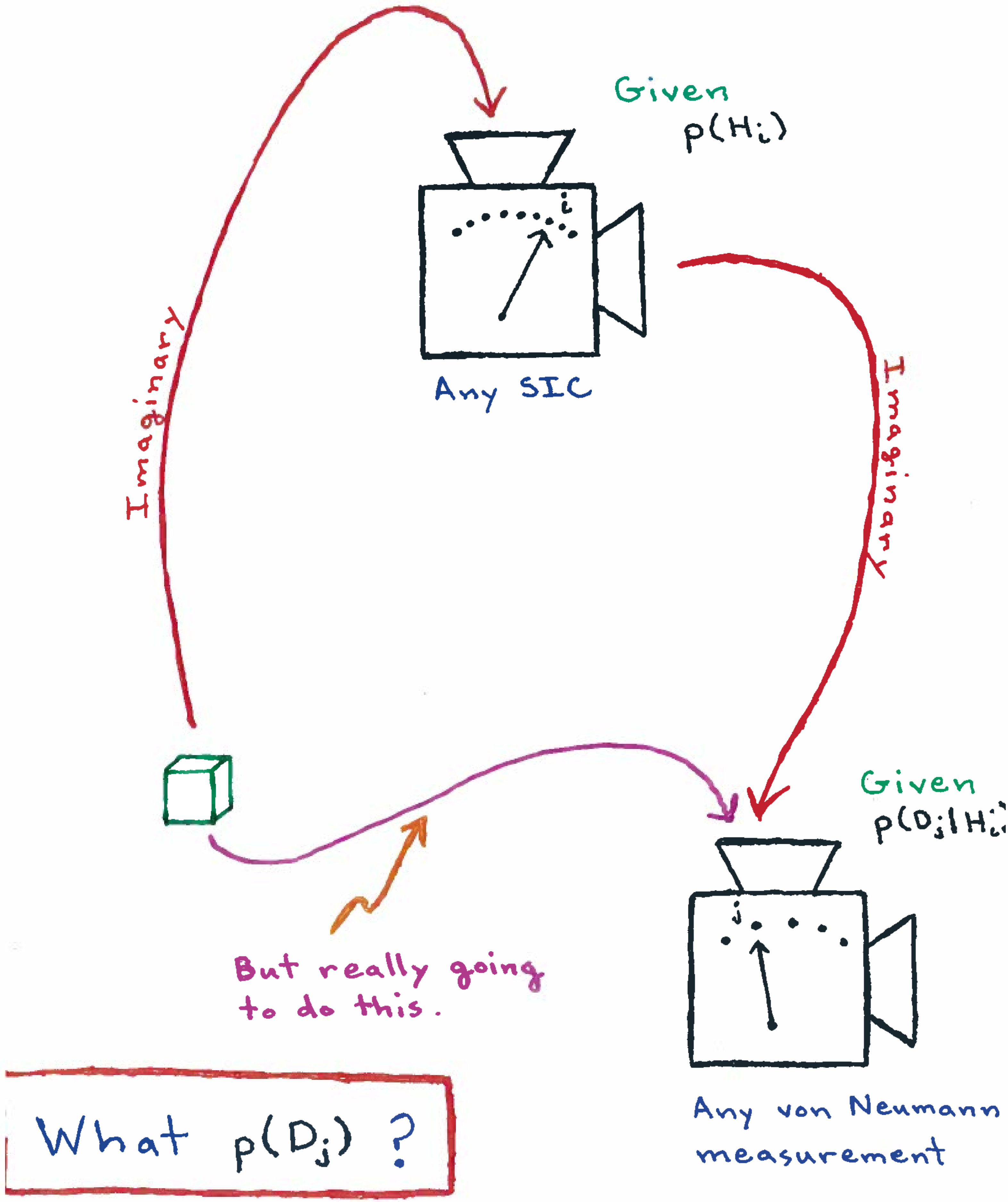
D_j – data values one might gather

Given: $p(D_j | H_i)$ ← expectations for data given hypothesis

$p(H_i)$ ← expectations for hypotheses themselves

Question: What expectations should one have for the D_j?

Answer: $P(D_j) = \sum_i p(H_i) p(D_j | H_i)$


Given
p(H_i)
Any SIC
Imaginary
Imaginary
Given
p(D_j|H_i)
But really going to do this.
Any von Neumann measurement
What p(D_j) ?

$$\underbrace{p(D_j)}_{\text{Quantum}} = (d+1) \underbrace{\sum_i p(H_i)\, p(D_j|H_i)}_{\text{(Usual) Bayesian}} - 1$$

Magic!

Generalizations

When measurement on the ground is any other SIC:

$$p(D_j) = (d+1)\sum_i p(H_i)\,p(D_j|H_i) - \frac{1}{d}$$

(Compare to unitary evolution.)

And

When measurement on the ground is a completely general POVM $\{D_j\}$, $j=1,\ldots,m$,

$$\boxed{p(D_j) = (d+1)\sum_i p(H_i)\,p(D_j|H_i) - \frac{1}{d}\sum_i p(D_j|H_i)}$$

Unitarity

$$\begin{tikzcd} \rho \arrow[r] \arrow[d] & U\rho U^\dagger \arrow[d] \\ p(i) \arrow[r] & q(j) \end{tikzcd}$$

Define $t(j|i) = \frac{1}{d}\,\mathrm{tr}\, U\Pi_i U^\dagger \Pi_j$

doubly stochastic matrix

Then

$$\boxed{q(j) = (d+1)\sum_i p(i)\,t(j|i) - \frac{1}{d}}$$

Could hardly be simpler.

Bayesian Perspective

No logical reason why situation with conditional lotteries should be commensurate with situation without conditional lotteries.

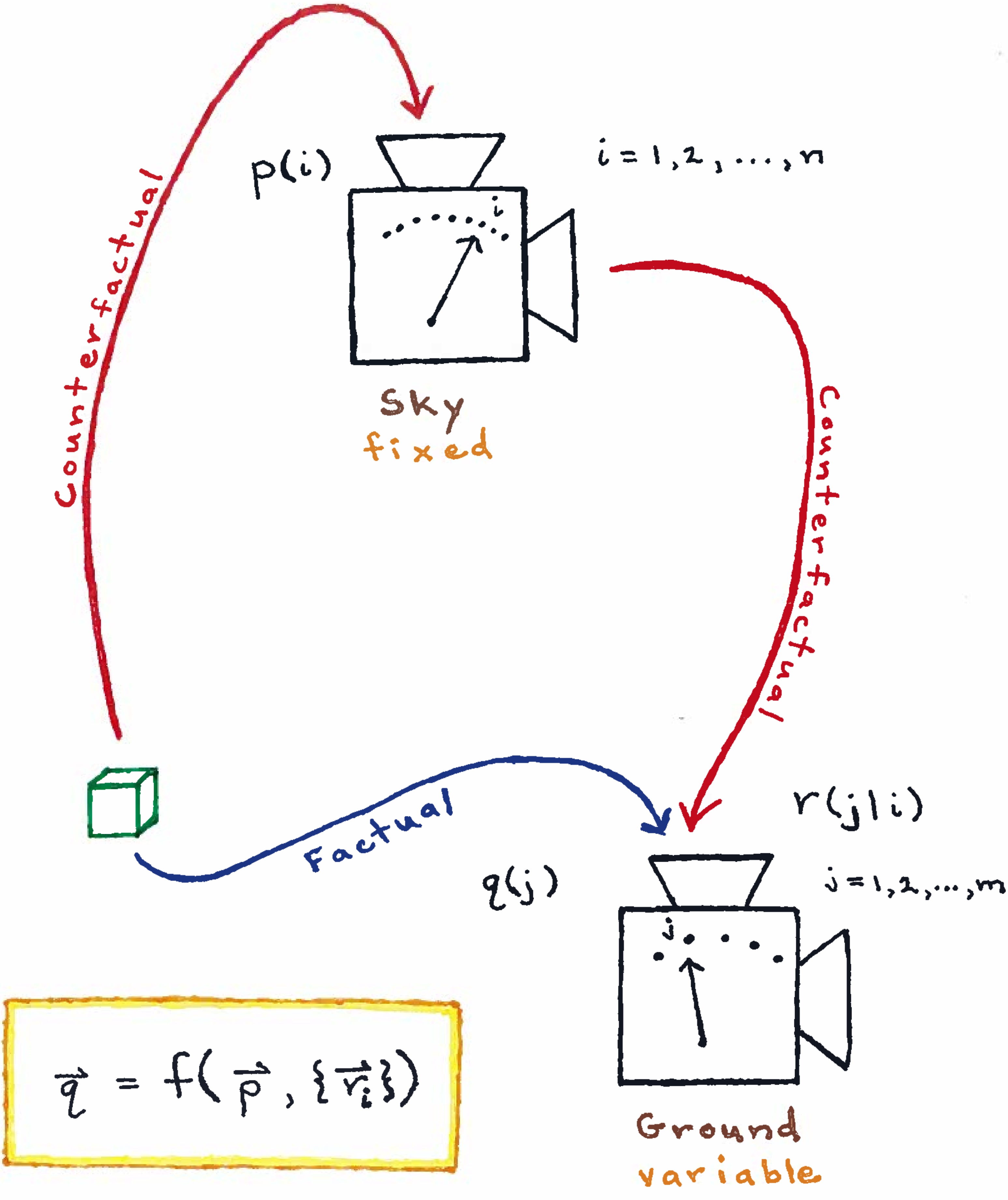
$$p(D_j) \neq \sum_i p(H_i)\,p(D_j|H_i)$$

(Need better notation, though.)

Quantum Perspective

Nonetheless, there may be empirical reasons for adopting a relation.

This is the content of the Born Rule.


Counterfactual
p(i)
i = 1, 2, ..., n
i
Sky
fixed
Counterfactual
Factual
r(j|i)
q(j)
j = 1, 2, ..., m
j
Ground
variable
$\vec{q} = f(\vec{p}, \{\vec{r_i}\})$

Fundamental Postulate: For any measurement on the ground, we can always calculate

$$q(j) = \alpha \sum_{i=1}^{n} p(i) r(j|i) - \beta \sum_{i=1}^{n} r(j|i)$$

$$\forall\ j = 1, \ldots, m; \quad \alpha, \beta > 0 \text{ fixed}.$$

Normalization requires $n\beta = \alpha - 1$.

Consequently, all valid $\vec{p}$ and $\vec{r}_i$ must satisfy

$$0 \leq \alpha \sum_{i} p(i) r(j|i) - \beta \sum_{i} r(j|i) \leq 1$$

the "urungleichung"

What further postulates must be made to recover precisely quantum state space?

I.e. the convex hull of

1) $\sum_i p(i)^2 = \frac{2}{d(d+1)}$

2) $\sum_{ijk} c_{ijk}\, p(i)\, p(j)\, p(k) = \frac{d+7}{(d+1)^3}$

with c_{ijk} possessing correct properties

Hints of Austerity

1) Mild postulate: $p(i) = \frac{1}{n}$ valid.

2) Principle of Reciprocity

Promote counterfactual to factual and suppose $p(i) = \frac{1}{n}$.

Postulate: Then posterior for i,

$$\mathrm{Prob}(i|j) = \frac{r(j|i)}{\sum_k r(j|k)}$$

is a valid prior; all priors like this.

3) Measurements for which I can have certainty for outcomes j given appropriate prior $p_j(i)$. ← Postulate

$$\Longrightarrow \quad m \leq \frac{\beta n}{\alpha - \beta n}$$

4) #2 already implies $\sum_i p(i)^2 \leq \frac{1}{\alpha}\left(\frac{m}{n} + \beta\right)$.

Think SIC thoughts!

... and maybe by way of it we'll come to understand quantum mechanics a little better.