Entanglement, area law and group theory

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Abstract.
We investigate bipartite entanglement in a class of multiparty quantum states, the G-states, constructed from a group G, and we derive an expression for the von Neumann entropy. We show that for a special subset of such states, the G-homogeneous states, the entropy satisfies the area law. If G is a group of spin-flips, the G-homogeneous states are locally equivalent to 2-colorable graph states (e.g., n-GHZ, cluster states, etc). The advantage of our representation is a more compact description of these states in terms of a group of spin-flips G. As an example, we compute the n-tangle τ_n for 2-colorable graph states.

Keywords. Bipartite entanglement; stabilizer/graph states; area law.

1. Introduction

Entanglement emerged recently as an important tool in the study of several fields, including quantum information processing, lattice spin systems and quantum phase transitions. Despite the considerable effort put into classifying and quantifying entanglement for many-body systems, there is no known measure which completely characterizes the entanglement properties of an arbitrary system.

In this article we investigate bipartite entanglement using a group theoretical framework, giving an overview of the results and methods discussed in [1,2]. The main role will be played by a set of quantum states, the G-states, defined in terms of a group G acting on the Hilbert space of the system. We derive a general expression for the von Neumann entropy corresponding to a bipartition (A, B) of the full system. For a special class of states, the G-homogeneous states, we show that the entropy satisfies the area law. If G is a group of spin-flips, the G-homogeneous states are locally equivalent to a well-known class of entangled states, the 2-colorable graph states, which include n-GHZ and cluster states. Finally, using this new representation we obtain a simple formula for the n-tangle τ_n of an arbitrary 2-colorable graph state.

2. Entanglement, G-states and G-homogeneous states

Let $\mathcal{H}$ be the Hilbert space of a quantum many-body system. In order to study bipartite entanglement in this system, we assume that we can partition it into two subsystems, A and B, such that $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. In the case of a spin system, this corresponds to take A as a particular set of spins and B the complementary set. Our main tool will be the *entanglement entropy* (von Neumann entropy) defined as $S = -\mathrm{Tr}\,(\rho_A \log_2 \rho_A)$, where ρ_A is the reduced density matrix of the A subsystem.

The focus will be on a class of quantum states defined in terms of a group. Let G be a group having a bilocal action on $\mathcal{H}$, i.e., $\forall g \in G,\ \ g = g_A \otimes g_B$, where $g_{A,B}$ acts only on $\mathcal{H}_{A,B}$, respectively. Let $|0\rangle = |0_A\rangle \otimes |0_B\rangle \in \mathcal{H}$ be a reference product state; for a n-spins system we can take the reference state to be the fully polarized state $|0\rangle = |0\rangle_1 \otimes \ldots |0\rangle_n$. With these premises, we introduce a class of (normalized) states, the G-states, defined as

$$|\Psi_G\rangle := \sum_{g\in G} \alpha(g) g|0\rangle \tag{1}$$

A particularly important subset of the above class are the G-homogeneous states, for which all the coefficients are equal:

$$|G\rangle := \mathcal{N}^{-1/2} \sum_{g\in G} g|0\rangle \tag{2}$$

where $\mathcal{N} \neq 0$ is a normalizing factor. We point out an important property of G-homogeneous states, namely that they are *stabilized* by the group G, i.e., $g|G\rangle = |G\rangle,\ \forall g \in G$.

Consider now the two subgroups of G that act exclusively on the subsystems A and respectively B (the "local" groups):

$$G_A := \{g \in G \mid \ g = g_A \otimes \mathbb{1}_B\} \tag{3}$$

$$G_B := \{g \in G \mid \ g = \mathbb{1}_A \otimes g_B\} \tag{4}$$

As these subgroups are *normal*, we can define the quotient group (the "non-local" group)

$$G_{AB} := \frac{G}{G_A \times G_B} \tag{5}$$

This shows that we can partition G in equivalence classes $[h] \in G_{AB}$ as

$$G = \bigcup_{[h]\in G_{AB}} \{(g_A \otimes g_B)h \mid g_A \otimes \mathbb{1} \in G_A,\ \mathbb{1} \otimes g_B \in G_B\} \tag{6}$$

and inside each equivalence class the elements are parametrized by elements of the "local" group $G_A \times G_B$.

Theorem. Suppose we have a quantum system described by a G-state (1). Assume that the following separability condition holds for the coefficients $\alpha(g)$

$$\forall g \in G, \quad \alpha(g) \equiv \alpha(g_A \otimes g_B h) = \alpha_A(g_A)\alpha_B(g_B)\beta(h) \tag{7}$$

where $g_A \otimes \mathbb{1} \in G_A,\ \mathbb{1} \otimes g_B \in G_B$ and $[h] \in G_{AB}$; without loss of generality, we can take $\beta(h)$ to be normalized, $\sum_h |\beta(h)|^2 = 1$. Then the von Neumann entropy of the G-state corresponding to the bipartition (A, B) is bounded by

$$S(|\Psi_G\rangle) \leq -\sum_{[h]\in G_{AB}} |\beta(h)|^2 \log_2 |\beta(h)|^2 \tag{8}$$

For a proof we refer the reader to [1].
Corollary 1. The von Neumann entropy for a G-homogeneous state satisfies

$$S(|G\rangle) \leq \log_2 |G_{AB}| \tag{9}$$

Observation 1. The bounds in eqs.(8),(9) are saturated if the projections on the A and B subsystems of all (nontrivial) group elements (i.e., $g_{A,B}$) have zero expectation value on the reference state [1]:

$$g_x \neq \mathbb{1}_x \Rightarrow \langle 0_x|g_x|0_x\rangle = 0, \qquad x = A, B \tag{10}$$

We now discuss the significance of this result. The "nonlocal" group G_{AB} is generated by operators acting on both subsystems which cannot be factorized in unitaries of G acting only on A or B subsystems. Suppose we have a multi-partite quantum system whose degrees of freedom have nearest neighbor interaction and let A be the set of all the degrees of freedom inside a closed surface. Then the "local" groups G_A and G_B correspond to the bulk degrees of freedom in the two subsystems, whereas G_{AB} corresponds to the degrees of freedom localized on the boundary between A and B. Hence we can interpret eq. (9) as another manifestation of the so-called area law (or more generally, the Holographic Principle): *the entropy of a system is proportional to the boundary degrees of freedom and not to the bulk ones*. The generic behaviour expressed by the area law has been recovered in several (apparently unconnected) systems, like black holes, scalar fields on a lattice [3] and harmonic oscillators [4], to name only a few. For a group of spin flips (see next section), we have $|G_{AB}| = 2^{n_{AB}}$, where n_{AB} is the number of independent generators of G_{AB}. Then from eq.(9) the entropy is $S = n_{AB} \leq \Sigma$ and is bounded by the area Σ of the surface; here we can take as a measure of area the number of "punctures" a generator makes in the boundary, and hence $n_{AB} \leq \Sigma$.

From eq.(8) we can derive immediately a sufficient condition for $|\Psi_G\rangle$ to be separable with respect to the (A, B) partition.
Corollary 2. If the group is a direct product $G = G_A \times G_B$, then $S(|\Psi_G\rangle) = 0$.

3. G-homogeneous states for a group of spin flips

In the formalism discussed so far, G was an arbitrary, possibly non-Abelian, group having a bilocal action on $\mathcal{H}$. We now turn to a case of special interest, namely we take G to be a group of spin flips. In the following we will use the notation X_i, Y_i, Z_i for the (generalized) Pauli operators acting on the i-qubit (qudit). First we notice that an arbitrary state of a n-qubit system $|\psi\rangle \in \mathcal{H}$ has the form (1). Let $\{|i_1 \dots i_n\rangle\}$, $i_k = 0, 1$, be the computational basis of $\mathcal{H}$. Then $|i_1 \dots i_n\rangle = \prod_{k=1}^{n} X_k^{i_k}|0\rangle$, and therefore the group is $G = N_1^{\otimes n}$, where $N_1 = \{\mathbb{1}, X\}$ is the group generated by a single spin-flip. Obviously, in this case $G \cong \mathbb{Z}_2^{\otimes n}$. The condition (10) is automatically satisfied, as $\langle 0|\prod_i X_i|0\rangle = 0$ for any product of spin-flip operators. [1]

In this case the associated G-homogeneous state is a *stabilizer state*. An n-qubit stabilizer state is invariant under the action of a group (the stabilizer) having n generators.

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[1] This can be generalized to an arbitrary system of qudits, since the computational basis of a single qudit Hilbert space is generated by applying the generalized Pauli x-operator on $|0\rangle$: $X|i\rangle = |i \oplus 1\rangle$, with $\oplus$ addition modulo *d*. The group is $G \cong \mathbb{Z}_d^{\otimes n}$.

In the following we will discuss the relationship between G-homogeneous states and stabilizer states. The connection will be made via graph states [5,6], as it has been shown that any stabilizer state is equivalent to a graph state under local Clifford operations [7].

Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}_2^n,\ a_k = 0, 1$, be a binary n-vector. We can write an arbitrary spin flip operator on the n-qubit system as $X(\mathbf{a}) := \prod_{i=1}^n X_i^{a_i}$. The full group can be written as $G = X(\mathcal{A})$, where $\mathcal{A} \subset \mathbb{Z}_2^n$ is the group of binary vectors corresponding to all the elements of G; the group operation of $\mathcal{A}$ is addition modulo 2. We define the inner product of two binary vectors as $(\mathbf{a}, \mathbf{b}) := \sum_{i=1}^n a_i b_i$ mod 2. Then one can prove that the stabilizer of $|G\rangle$ is [1]:

$$\mathcal{S}_{|G\rangle} = X(\mathcal{A}) \cdot Z(\mathcal{A}^{\perp}) \tag{11}$$

where $\mathcal{A}^{\perp} = \{\mathbf{b} \mid (\mathbf{a}, \mathbf{b}) = 0,\ \forall \mathbf{a} \in \mathcal{A}\}$ is the orthogonal complement of $\mathcal{A}$. Since the stabilizer generators are only of X- or Z-type, the G-homogeneous state is a Calderbank-Shor-Steane (CSS) state [8], which are locally equivalent to 2-colorable graph states [9,1]. Therefore G-homogeneous states are locally equivalent to 2-colorable graph states.

Since $\mathcal{A}$ and $\mathcal{A}^{\perp}$ are dual, $\dim \mathcal{A} + \dim \mathcal{A}^{\perp} = n$, we can choose the group G to have at most $\lfloor n/2 \rfloor$ generators ($\dim \mathcal{A}$ is the number of its generators). This can be readily seen from the following argument. Define $|G'\rangle := H^{\otimes n}|G\rangle$, where H is a Hadamard gate; obviously the two states are locally equivalent, $|G'\rangle \simeq_{LU} |G\rangle$. Since the Hadamard gate interchanges the X and Z operators, the stabilizer of the new state is $\mathcal{S}_{|G'\rangle} = X(\mathcal{A}^{\perp}) \cdot Z(\mathcal{A})$, hence the new spin-flip group is $G' = X(\mathcal{A}^{\perp})$. Therefore, for every G-homogeneous state $|G\rangle$, with G having k generators, there is a locally equivalent G'-homogeneous state $|G'\rangle$, where G' has now $n - k$ generators. As both $|G\rangle$ and $|G'\rangle$ have the same entanglement, one can immediately derive a bound for the entropy of G-homogeneous states:

$$S(|G\rangle) \leq \min(k, n-k) \leq \lfloor n/2 \rfloor \tag{12}$$

The compact representation of 2-colorable graph states as G-homogeneous states (modulo local unitaries) enables us to easily compute a measure of n-partite entanglement, the n-tangle. The n-tangle τ_n is defined, for n even, as $\tau_n := |\langle\psi|Y^{\otimes n}|\psi^*\rangle|^2$ [10]. A simple computation for a G-homogeneous state (and hence for a 2-colorable graph state) gives for the n-tangle the following result [1]:

$$\tau_n(|G\rangle) = (1 - p(G))\ \chi_G(X^{\otimes n}) \tag{13}$$

where $p(G) = 0$ if all generators of G are even (i.e., are a product of an even number of spin-flip operators) and $p(G) = 1$ otherwise; χ_G is the characteristic function of G. Hence the n-tangle is 1 iff all the generators of G are even and $X^{\otimes n} \in G$.

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