What is a probability monad?

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Categorical Probability 2020
Tutorial video
Monads as extensions

Definition:
Let $C$ be a category. A monad on $C$ consists of:
- A functor $T : C \to C$;
- A natural transformation $\eta : \text{id}_C \Rightarrow T$ called unit;
- A natural transformation $\mu : TT \Rightarrow T$ called composition;

such that the following diagrams commute:
Monads as extensions

Idea:
A monad is like a consistent way of extending spaces to include generalized elements and generalized functions of a specific kind.
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1. To each space $X$, an “extended” space $TX$. 
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2. Given $f : X \rightarrow Y$, an “extension” $Tf : TX \rightarrow TY$. 
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Monads as extensions
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\[ x_1 \quad x_2 \quad x_3 \]

\[ X \quad TX \]

- \( x_1 \)
- \( x_2 \)
- \( x_3 \)

\( x_1 \cdot x_2 \cdot x_3 \)
A natural transformation $\eta : \text{id}_C \Rightarrow T$ consists of:

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Monads as extensions

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$$
\begin{array}{cccc}
X & \xrightarrow{f} & Y \\
\downarrow{\eta_X} & & \downarrow{\eta_Y} \\
TX & \xrightarrow{Tf} & TY
\end{array}
$$
Monads as extensions

A natural transformation $\mu : TT \Rightarrow T$, is:
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2. Again a naturality diagram as before.
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\text{Diagram:}
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**Definition:**
Let $T$ be a monad on $C$. A *Kleisli morphism* from $X$ to $Y$ is a morphism $X \to TY$. 
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![Diagram of monads as extensions]

$X$ $Y$
Monads as extensions

Definition:
Given Kleisli morphisms $k : X \rightarrow TY$ and $h : Y \rightarrow TZ$, their Kleisli composition is the morphism $h \circ_{kl} k$ given by:

$$X \xrightarrow{k} TY \xrightarrow{Th} TTZ \xrightarrow{\mu} TZ$$
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![Diagram showing the composition of Kleisli morphisms]

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$$X \xrightarrow{k} \bullet \quad Y \xrightarrow{Th} \bullet \quad Z$$
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![Diagram showing the composition of Kleisli morphisms]

$X$ $Y$ $Z$
Monads as extensions

Exercise:
Prove that Kleisli morphisms form a category thanks to the commutativity of these diagrams:

\[
\begin{align*}
TX & \xrightarrow{T\eta} TTX \\
& \Downarrow \mu \\
& TX
\end{align*}
\]

\[
\begin{align*}
TX & \xrightarrow{\eta T} TTX \\
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where the identity morphisms of the Kleisli category are given by the units \( \eta : X \to TX \).
Probability monads

Idea [Giry, 1982]:
Spaces of “random elements” generalizing usual elements.
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- Base category C
- Functor \( X \mapsto PX \)
- Unit \( \delta : X \to PX \)
- Composition \( E : PPX \to PX \)
A Kleisli morphism from $X$ to $Y$ is a morphism $X \rightarrow PY$. We can interpret this as a “random function” or “random transition”.
Probability monads

Given Kleisli morphisms \( k : X \to PY \) and \( h : Y \to PZ \), their Kleisli composition is the morphism \( h \circ_{kl} k \) given by:

\[
\begin{align*}
X & \xrightarrow{k} PY & \xrightarrow{Ph} PPZ & \xrightarrow{E} PZ
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\[
\begin{array}{c}
X \\
\bullet
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\[ X \xrightarrow{k} PY \xrightarrow{Ph} PPZ \xrightarrow{E} PZ \]
The distribution monad on Set

Definition:
Let $X$ be a set. A f.s. distribution on $X$ is a function $p : X \to [0, 1]$ such that
- It is nonzero for finitely many $x \in X$;
- $\sum_{x \in X} p(x) = 1$.
We denote by $DX$ the set of f.s. distributions on $X$. 
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![Diagram of X with points]
The distribution monad on Set

**Definition:**
Let \( f : X \to Y \) be a function and \( p \in DX \). The pushforward of \( p \) along \( f \) is the distribution \( f_*p \in DY \) given by

\[
f_*p(y) := \sum_{x \in f^{-1}(y)} p(x).
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We denote the map \( f_* : DX \to DY \) by \( Df \), this makes \( D \) a functor.
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**Definition:**
Let $X$ be a set. The map $\delta : X \to DX$ maps $x \in X$ to the distribution $\delta_x \in DX$ given by

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\delta_x(y) = \begin{cases} 
1 & y = x; \\
0 & y \neq x.
\end{cases}
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This gives a natural map $\delta : X \to DX$, a component of the unit of the monad.
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**Definition:**
Let \( X \) be a set. Given \( \xi \in DDX \), define \( E\xi \in DX \) to be distribution given by

\[
E\xi(x) := \sum_{p \in DX} p(x) \xi(p).
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This gives a natural map \( E : DDX \to DX \), a component of the multiplication of the monad.
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The distribution monad on Set

**Kleisli morphisms:**

A Kleisli morphism for $D$ is a function $k : X \rightarrow DY$. In other words, it is function $\bar{k} : X \times Y \rightarrow [0, 1]$ such that

- For each $x \in X$, $\bar{k}(x, -) : Y \rightarrow [0, 1]$ is nonzero in finitely many entries;
- For each $x \in X$, $\sum_{y \in Y} \bar{k}(x, y) = 1$. 
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Kleisli composition:
The Kleisli composition of $k : X \rightarrow DY$ and $h : Y \rightarrow DZ$ is given by the Chapman-Kolmogorov equation:

$$(h \circ_{kl} k)(x, z) = \sum_{y \in Y} k(x, y) h(y, z).$$
The Giry monad on Meas

Let $X$ be a measurable space. Define $PX$ to be

- The set of probability measures on $X$
The Giry monad on Meas

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- Equipped with the $\sigma$-algebra generated by the evaluation functions
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- Equipped with the $\sigma$-algebra generated by the evaluation functions $\varepsilon_A : PX \to \mathbb{R}$ given by
  \[ p \mapsto p(A) \]
  for all $A \subseteq X$ measurable.
- Equivalently, the $\sigma$-algebra is generated by the “integration” functions $\varepsilon_f : PX \to \mathbb{R}$ given by
  \[ p \mapsto \int f \, dp, \]
  for all $f : X \to [0, 1]$ measurable.
The Giry monad on Meas

**Functoriality:**
Let $f : X \to Y$ be a measurable function. Given a measure $p \in PX$, recall that the pushforward measure $f_*p \in PY$ is given by

$$f_*p(B) := p(f^{-1}(B)).$$

We get a measurable map $Pf : PX \to PY$ which makes $P$ a functor.
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We get a measurable map $Pf : PX \to PY$ which makes $P$ a functor.

Unit:
Given a measurable space $X$, to each $x \in X$ we can give the Dirac delta measure $\delta_x \in PX$. This gives a measurable map $\delta : X \to PX$, which is natural, and forms a component of the unit of the monad.
The Giry monad on Meas

**Multiplication:**
Given a measurable space \( X \) and a measure \( \pi \in PPX \), we define the measure \( E\pi \in PX \) by

\[
E\pi(A) := \int_{PX} p(A) d\pi(p),
\]

This gives a measurable map \( E : PPX \to PX \) which is natural in \( X \) and forms a component of the monad multiplication.
The Giry monad on Meas

Kleisli morphisms:
A Kleisli morphism is a measurable map $k : X \to PY$, in other words, a Markov kernel between $X$ and $Y$. Denote $k(x) \in PY$ by $k_x$. 
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Kleisli morphisms:
A Kleisli morphism is a measurable map $k : X \rightarrow PY$, in other words, a Markov kernel between $X$ and $Y$. Denote $k(x) \in PY$ by $k_x$.

Kleisli composition:
The composition of Kleisli morphisms reproduces the Chapman-Kolmogorov equation for general measures. Given $k : X \rightarrow PY$ and $h : Y \rightarrow PZ$, we get that

$$(h \circ_\mathcal{K} k)(x)(C) = \int_Y h_y(C) dk_x(y)$$

for each $x \in X$ and for each $C \subseteq Z$ measurable.
### Other probability monads

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More on the nLab, “probability monad” [nLab article].

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Joints and marginals

Idea:
Probability theory is mostly about *interactions* of random variables.

- Composite states
  \( X \times Y \)
Joints and marginals

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Probability theory is mostly about *interactions* of random variables.

- Composite states $X \times Y$
- Given marginals
Joints and marginals

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Probability theory is mostly about *interactions* of random variables.

- Composite states $X \times Y$
- Given marginals
- Many possible joints
Joints and marginals

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Joints and marginals

Idea:
Probability theory is mostly about *interactions* of random variables.

- Composite states $X \times Y$
- Given marginals
- Many possible joints
- One canonical choice of “independence”
Joints and marginals

Idea:
Given objects $X$ and $Y$, a probability distribution on $X \times Y$ is not just pair of distributions on $X$ and $Y$ separately. However, given $p \in PX$ and $q \in PY$, we get a measure $p \otimes q \in P(X \times Y)$.

$$PX \times PY \xrightarrow{\nabla} P(X \times Y)$$
Joints and marginals

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$$PX \times PY \xrightarrow{\nabla} P(X \times Y)$$

This gives a *monoidal structure* to the probability monad. (Technically, we need $\nabla$ together with a map $1 \rightarrow P1$, but for probability monads $1$ and $P1$ are uniquely isomorphic.)
Joints and marginals

\[ PX \times PY \times PZ \xrightarrow{\nabla \times \text{id}} P(X \times Y) \times PZ \]

\[ PX \times P(Y \times Z) \xrightarrow{\nabla} P(X \times Y \times Z) \]
Joints and marginals

\[ X \times Y \]

\[ \pi_1 \quad \pi_2 \]

\[ X \quad Y \]
Joints and marginals

\[ P(X \times Y) \]

\[ P_X \]

\[ P_Y \]

\[ P_{\pi_1} \]

\[ P_{\pi_2} \]
Joints and marginals

\[ P(X \times Y) \]

\[ \overset{P_{\pi_1}}{\text{PX}} \quad \overset{P_{\pi_2}}{\text{PY}} \]

\[ \overset{\pi_1}{\text{PX}} \quad \overset{\pi_2}{\text{PY}} \]
Joints and marginals

\[ P(X \times Y) \]

\[ P(\pi_1) \quad P(\pi_2) \]

\[ PX \quad PX \times PY \quad PY \]

\[ \Delta \]

\[ P(X \times Y \times Z) \xrightarrow{\Delta \times \text{id}} PX \times P(Y \times Z) \]

\[ \xrightarrow{\text{id} \times \Delta} P(X \times Y) \times PZ \xrightarrow{\Delta} PX \times PY \times PZ \]
Joints and marginals

Operations on distributions:
Let \( f : X \times Y \rightarrow Z \) be a binary function. Then we can form the map

\[
PX \times PY \xrightarrow{\nabla} P(X \times Y) \xrightarrow{Pf} PZ
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Joints and marginals

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Let $f : X \times Y \rightarrow Z$ be a binary function. Then we can form the map

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For example, the addition as a map $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ gives the convolution of real-valued random variables.
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