Problem Set #1

Quantum Error Correction
Instructors: Daniel Gottesman and Beni Yoshida

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Problem #1. Bosonic codes

A bosonic mode is an infinite-dimensional Hilbert space with a standard basis labelled by the non-negative integers, i.e., $|0\rangle, |1\rangle, |2\rangle, |3\rangle, \ldots$, representing the eigenstates of a harmonic oscillator. For instance, for light, $|j\rangle$ is a state with j photons in this mode (the mode specifying a particular wavenumber and spatial profile). For bosonic modes, there is a natural generalization of the amplitude damping channel to

$$\rho \mapsto \sum_k A_k \rho A_k^\dagger, \tag{1}$$

with

$$A_k = \sum_{j \geq k} \sqrt{\binom{j}{k}} \sqrt{(1-\gamma)^{j-k}\gamma^k}\ |j-k\rangle\langle j|, \tag{2}$$

representing loss of k photons from a mode. γ indicates the rate of photon loss. In particular,

$$A_0 = \sum_j (1-\gamma)^{j/2}\ |j\rangle\langle j| \tag{3}$$

$$A_1 = \sum_{j \geq 1} \sqrt{j(1-\gamma)^{j-1}\gamma}\ |j-1\rangle\langle j|. \tag{4}$$

Note that, as with amplitude damping, A_0 is not proportional to the identity — more highly excited states are more likely to emit photons, so not having a photon loss event makes it more likely there were fewer photons to begin with.

For this problem, we will look at codes encoding a single qubit in n bosonic modes to correct for loss of a single photon from one mode. Let $B_0 = A_0^{\otimes n}$ be the no-loss operator and $B_i = A_0^{\otimes i-1} \otimes A_1 \otimes A_0^{\otimes n-i}$ be the operator which has loss of 1 photon from the ith mode and no loss from the other modes. The error set that we are trying to correct is thus $\mathcal{E} = \{B_0, B_1, \ldots B_n\}$.

This problem has 3 parts (the third being on the back of the page).

a) Consider the following encoding:

$$|\bar{0}\rangle = \frac{1}{\sqrt{2}}(|40\rangle + |04\rangle) \tag{5}$$

$$|\bar{1}\rangle = |22\rangle. \tag{6}$$

Show that this is a QECC correcting the error set $\mathcal{E}$ for two modes.

b) Consider the following encoding:

$$|\bar{0}\rangle = \frac{1}{\sqrt{3}}(|300\rangle + |030\rangle + |003\rangle) \tag{7}$$

$$|\bar{1}\rangle = |111\rangle. \tag{8}$$

Show that this is a QECC correcting the error set $\mathcal{E}$ for three modes.

c) The total photon number of a multimode basis state $|j_1 j_2 \ldots j_n\rangle$ is $\sum_i j_i$. The total photon number of a superposition is only defined if all terms in the superposition have the same total photon number, and is then equal to that value. Thus, the codewords for the code in part a has total photon number 4 and the code in part b has total photon number 3. Show that there is no bosonic code for any number of modes that has total photon number 1.