Quantum Foundations,
Asher Peres Style

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Perimeter Inst.
| $\psi$ |
Quantum Mechanics

1) States

\[ |\psi\rangle = \sum_\alpha a_\alpha |e_\alpha\rangle \]

2) Evolutions

\[ |\psi\rangle = U |\psi\rangle \]

Unitary
(Schrödinger)

\[ U^{-1} = U^\dagger \]
More QM

3) Measurement

\[ \rho(b | \psi) = |\langle e'_b | \psi \rangle|^2 \]

4) Composition

\[ |\psi \rangle = \sum_{\alpha \beta} c_{\alpha \beta} |e_\alpha \rangle |f_\beta \rangle = \begin{cases} \text{entangled} \\ \text{if not} \\ |\psi \rangle |\lambda \rangle \end{cases} \]
“A wavefunction is not something which ‘exists’ in nature.”

— A. Peres

Example: The Qubit

A two-level atom or a photon's polarization

The spin of an electron

Schrödinger's cat

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

$$\Pi = |\psi\rangle\langle\psi| = |\alpha|^2 |0\rangle\langle 0| + \alpha\beta^* |0\rangle\langle 1| + \alpha\beta |1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|$$
The hypothesis that there is an external world, not dependent on human minds, made of something, is so obviously useful and so strongly confirmed by experience down through the ages that we can say without exaggerating that it is better confirmed than any other empirical hypothesis.

— Martin Gardner
Information/knowledge about what?

... the consequences of our experimental interventions into the course of Nature.
Notation Warning

Sloppily call both \( |\psi\rangle \) and \( \Pi = |\psi\rangle \langle \psi| \) "the quantum state".

But that's the right one.

\[
\text{tr } A\Pi = \text{tr } [A \vert\psi\rangle \langle \psi\vert] = \langle \psi | A | \psi \rangle
\]
When \( A = |\varphi\rangle \langle \varphi| \),

\[
\text{tr } A\Pi = \langle \psi | \varphi\rangle \langle \varphi | \psi \rangle = |\langle \psi | \varphi \rangle|^2.
\]
What counts as a quantum measurement?

And why?
von Neumann Measurements

"measurement"

\[ \sigma = \sum_i \alpha_i \Pi_i \]

eigenvalues eigenprojectors

When state is \( \rho \),

\[ p(i) = \text{tr} \rho \Pi_i \]

Could say,

"measurement" \( \iff \{ \Pi_i \} \)
"Measurement"

Theoretical Description
**Bloch Sphere**

\[ \rho = |\psi\rangle\langle\psi| \in \mathcal{L}(\mathcal{H}_2) \text{ i.e., qubit} \]

\[
\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

\[
I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
\rho = \frac{1}{2}(I + a_x \sigma_x + a_y \sigma_y + a_z \sigma_z)
\]

\[
= \frac{1}{2} \begin{bmatrix} 1 + a_z & a_x + ia_y \\ a_x - ia_y & 1 - a_z \end{bmatrix}
\]

\[
\vec{a} = (a_x, a_y, a_z) \text{ with } |\vec{a}|^2 = 1.
\]

**Theorem** \( \text{tr} \rho \Pi = 0 \) iff \( \vec{a} = -\vec{b} \).
POVMs

Positive Operator Valued Measures

- an immensely useful tool

Let \( \mathcal{P} = \{ E : 0 \leq \langle \psi | E | \psi \rangle \leq 1 \ \forall | \psi \rangle \} \).

Any set of operators

\[ \{ E_b : E_b \in \mathcal{P}, \sum_b E_b = I \} \]

corresponds to a potential mmt.

Probability of outcome \( b \),

\[ p_b = \text{tr} \rho E_b. \]
Generalized Measurements

or **POVMs** — positive operator-valued measures

1) **Couple** system to be "measured" to some "ancilla".

2) Let the two interact unitarily.

3) Perform standard measurement on ancilla.

\[ \rho(b) = \text{tr} \hat{\rho} \hat{E}_b \]

with \( \langle \psi | \hat{E}_b | \psi \rangle > 0 \ \forall | \psi \rangle \)

and \( \sum \hat{E}_b = \hat{1} \)
Dirac Notation

Tensor Product Space:

For two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ define $\mathcal{H}_1 \otimes \mathcal{H}_2$ to be composed of all ordered pairs $|\psi_1\rangle |\psi_2\rangle$, where $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$, and all linear superpositions thereof.

Compatibility requirements —

\[
(\alpha |\psi_1\rangle + \beta |\psi_2\rangle) |\varphi\rangle = (\alpha |\psi_1\rangle |\varphi\rangle + \beta |\psi_2\rangle |\varphi\rangle \\
= |\psi_1\rangle (\alpha |\varphi\rangle) + |\psi_2\rangle (\beta |\varphi\rangle)
\]

\[
\text{IP} (|\psi_1\rangle |\psi_2\rangle , |\varphi_1\rangle |\varphi_2\rangle) = \langle \psi_1 |\varphi_1\rangle \langle \psi_2 |\varphi_2\rangle
\]

inner product

etc.
**Dirac Notation.**

Operators in $\mathcal{O}(\mathcal{H}_1 \otimes \mathcal{H}_2)$:

If $|e_i\rangle$ and $|e_j\rangle$ are orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then $|e_i\rangle\langle e_j|\otimes \langle e'_k|\otimes |e'_l\rangle$ is one for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Thus any $A \in \mathcal{O}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ can be written

$$A = \sum_{ijkl} A_{ijkl} |e_i\rangle\langle e_j|\otimes \langle e'_k|\otimes |e'_l\rangle.$$ 

---

**Partial trace:**

$$\text{tr}_1 A = \sum_{ik} A_{ik} |e_i\rangle\langle e_k|\otimes |e'_l\rangle$$

$$\text{tr}_2 A = \sum_{ik} A_{ik} |e_i\rangle\langle e'_k|\otimes |e'_l\rangle.$$
1) Couple system to be measured to some "ancilla" : \( \rho_s \otimes \rho_A \)

2) Let the two interact unitarily: 
\[
\rho_s \otimes \rho_A \rightarrow U(\rho_s \otimes \rho_A)U^+ 
\]

3) Perform basic measurement \( \{p_1, \ldots, p_k\} \) on ancilla alone

4) Outcome \( b \) occurs with probability 
\[
p(b) = \text{tr} \left[ U(\rho_s \otimes \rho_A)U^+ (I \otimes p_b) \right]
\]
Where POVMs Come From

\[ \rho(b) = tr \left[ U(\rho_s \otimes \rho_A)U^+(I \otimes \rho_b) \right] \]
\[ = tr \left[ (\rho_s \otimes \rho_A)U^+(I \otimes \rho_b)u \right] \]
\[ = tr \left[ (\rho_s \otimes I)(I \otimes \rho_A)U^+(I \otimes \rho_b)u \right] \]
\[ = tr_s \left[ \rho_s \ tr(A \{ (I \otimes \rho_A)U^+(I \otimes \rho_b)u \}) \right] \]
\[ = tr \rho_s E_b \]

where

\[ E_b = tr_A \left[ (I \otimes \rho_A)U^+(I \otimes \rho_b)u \right] \]

POVM element
But why POVMs?

Needed!

POVM proportional to projectors

$E_x = \frac{2}{3} P_2$

$E_1 = \frac{2}{3} P_1$

$E_0 = \frac{2}{3} P_0$
But why POVMs?

Needed!
But why POVMs?

Needed!

Sometimes eliminates one, but not always.
But why \textbf{POVMs}?

Needed!

Always eliminates one.
Why not take POVMs as basic notion of measurement?
<table>
<thead>
<tr>
<th>Standard Measurements</th>
<th>Generalized Measurements</th>
</tr>
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<tbody>
<tr>
<td>${\pi_i}$</td>
<td>${E_b}$</td>
</tr>
<tr>
<td>$\langle \psi</td>
<td>\pi_i</td>
</tr>
<tr>
<td>$\sum_i \pi_i = I$</td>
<td>$\sum_b E_b = I$</td>
</tr>
<tr>
<td>$\rho(i) = tr \rho \pi_i$</td>
<td>$\rho(b) = tr \rho E_b$</td>
</tr>
<tr>
<td>$\pi_i \pi_j = \delta_{ij} \pi_i$</td>
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</table>

Does this extra assumption really make the process any less mysterious?
<table>
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Does that really make it any more mysterious?

?
Density Operators

$\rho \in \mathcal{L}(\mathcal{H})$

- Linear operators
- Complex vector space
- Catalog of uncertainties

1. $\rho^+ = \rho$
2. $\text{tr} \rho = 1$
3. $\lambda_i(\rho) \geq 0$

Convex hull of the set $\{ |\psi\rangle\langle\psi| : |\psi\rangle \in \mathcal{H} \}$
Gleason's Theorem

Let $\mathcal{P}(\mathcal{H}_d)$ be the set of 1-D projectors onto a (real or complex) vector space $\mathcal{H}_d$ of dimension $d \geq 3$.

Suppose there exists a function $f: \mathcal{P}(\mathcal{H}_d) \to [0,1]$ such that

$$\sum_i f(\Pi_i) = 1$$

whenever $\{\Pi_i\}$ forms a complete orthogonal set.

**Theorem**: Then there exists a density operator $\rho$, such that

$$f(\Pi) = \text{tr} \rho \Pi.$$

**Main Assumptions (my take)**:

1. **Measurements are incompatible.**
   No good notion of measuring $\{\Pi_i\}$ AND $\{\tilde{\Pi}_i\}$.

2. **Noncontextuality of probabilities.**
   $\text{Prob} \ (i \mid \{\Pi_k\}) = f(\Pi_i)$,
Comments

1) Proof is long, hard, ugly.
   a) Reduce problem to checking $\mathbb{R}^2$.
   b) Prove continuity of $f$.
   c) Expand $f$ in spherical harmonics and massage.

2) Doesn’t work for $d=2$.
   
   Prompts people to say “there’s nothing quantum about a single qubit.”

3) Gave Dave Meyer another PRL!
   
   Theorem fails for fields other than $\mathbb{C}$ and $\mathbb{R}$, like $\mathbb{Q}$ — the rationals.
Proof (for rational $\mathcal{P}$)

Consider $E_1, E_2 \in \mathcal{P}$ s.t. $E_1 + E_2 \in \mathcal{P}$. Embed in 3-outcome POVM.

$f(E_1) + f(E_2) + f(E_3) = 1$
$f(E_1 + E_2) + f(E_3) = 1$

$\Rightarrow f(E_1 + E_2) = f(E_1) + f(E_2)$

Similarly for integers $p, q$,

$f(E) = p \cdot f(\frac{1}{p}E) = q \cdot f(\frac{1}{q}E) \Rightarrow f(\frac{p}{q}E) = \frac{p}{q} f(E)$. etc.

Note $\mathcal{P}$ spans the space of operators.
Choose a complete basis $E_i \in \mathcal{P}$, $i = 1, \ldots, d^2$.

$f(E) = f(\sum_i \alpha_i E_i) = \sum_i \alpha_i f(E_i)$

Define $\rho$ to satisfy $d^2$ equations

$\text{tr} \rho E_i = f(E_i)$ numbers
and use linearity of trace.
Gleason-like Theorems

Assumptions

1) Measurements = POVMs
2) Noncontextuality
\[ \text{Prob} (E_i | \{E_k\}) = \text{Prob} (E_i | \{\tilde{E}_k\}) \]

when \( \{E_k\} \) and \( \{\tilde{E}_k\} \) share \( E_i \).

I.e. Let \( f : \rho \rightarrow [0,1] \) be such that
\[ \sum_b f(E_b) = 1 \quad \text{whenever} \quad \sum_b E_b = \mathbf{I}. \]

**Thm:** \( \exists \ \rho \), s.t. \( f(E) = \text{tr} \rho E \).

Payoff:

1) Works for \( d = 2 \).
2) Works for rational \( d \).
3) Proof easy.
Extreme Points

Characterization 1:
\[ \rho = |\psi\rangle\langle\psi| \]

Characterization 2:
\[ \rho \text{ is hermitian} \]
\[ \rho^2 = \rho \]
\[ \text{tr } \rho = 1 \]

Characterization 3:
\[ \rho \text{ is positive semi-definite} \]
\[ \text{tr } \rho = 1 \]
\[ \text{tr } \rho^2 = 1 \]
Remarkable Theorem

Jones & Linden, PRA 71 (2005)
Flammia, (unpub, 2004)

Let $A$ be Hermitian, $A^+ = A$.

Then, $A = |\psi\rangle\langle\psi|$ if and only if

$$\text{tr} \ A^2 = \text{tr} \ A^3 = 1.$$
Proof:

$a_i$ - eigenvalues of $A$

$\text{tr } A^2 = \sum_i a_i^2 = 1 \implies |a_i| \leq 1$

$1 - a_i \geq 0$

$0 = \text{tr } A^2 - \text{tr } A^3 = \sum_i a_i^2 (1 - a_i)$

$\implies a_i = 0$ or $1 - a_i = 0$

$\text{tr } A^2 = 1 \implies a_i = 1$ for one and only one $i$.

QED
The Weatherman

$p(h, d)$

$p(h)$
The Weatherman

\[ p(h) = \sum_d p(h,d) \]
\[ = \sum_d p(d)p(h|d) \]

Bayesian Updating

\[ p(h) \xrightarrow{d} p(h|d) \]
\[ p(h) = \sum_d p(d)p(h|d) \]

"measurement" is any I-know-not-what that induces such a transition.
The Weatherman

Is this transition a mystery physics should contend with?

Even so, what does it have to do with the weather?
A quantum state, being a summary of the observers' information about an individual physical system, changes both by dynamical laws and whenever the observer acquires new information about the system through the process of measurement. The existence of two laws for the evolution of the state vector becomes problematical only if it is believed that the state vector is an objective property of the system. If the state of a system is defined as a list of [experimental] propositions together with [their probabilities of occurrence], it is not surprising that after a measurement the state must be changed to be in accord with the new information. The "reduction of the wave packet" does take place in the consciousness of the observer, not because of any unique physical process which takes place there, but only because the state is a construct of the observer and not an objective property of the physical system.
Quantum Axiom 5:

Suppose $S_1 \otimes S_2$ consists of two noninteracting systems (e.g., space-like separated systems) and $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is of the form

$$|\psi\rangle = \sum_i \sqrt{p_i} |e_i\rangle |\psi_i\rangle$$

for some orthonormal set $|e_i\rangle \in \mathcal{H}_1$, and $|\psi_i\rangle \in \mathcal{H}_2$ with $\langle \psi_i | \psi_i \rangle = 1$. ($\sum_i p_i = 1$)

Then a measurement of $\{ |e_i\rangle \langle e_i | \}$ on $S_1$, revealing outcome $k$ leaves the observer in a maximal state of knowledge about $S_2$:

$$\text{state}(S_2 | k) = |\psi_k\rangle \langle \psi_k |.$$
The More Pure Einstein

**Granted:** “The individual system (before the measurement) has no definite value of $q$ (or $p$). The value of the measurement only arises in cooption with the unique probability which is given to it in view of the $\psi$-function only through the act of measurement itself.”

**Consider spatially separated systems** $S_1$ and $S_2$ initially attributed with an entangled quantum state $\psi_{12}$.

“Now it appears to me that one may speak of the real factual situation at $S_2$. . . . [O]n one supposition we should, in my opinion, absolutely hold fast: the real factual situation of the system $S_2$ is independent of what is done with $S_1$. . . . According to the type of measurement which I make of $S_1$, I get, however, a very different $\psi_2$ for $[S_2]$. . . . For the same real situation of $S_2$ it is possible therefore to find, according to one’s choice, different types of $\psi$-function.

If now [physicist B] accepts this consideration as valid, then [he] will have to give up his position that the $\psi$-function constitutes a complete description of a real factual situation. For in this case it would be impossible that two different types of $\psi$-functions could be coordinated with the identical factual situation of $S_2$.”
$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$

Let Alice measure $|\uparrow\rangle$, $|\downarrow\rangle$ basis. Bob's system will be in state $|\uparrow\rangle$ or $|\downarrow\rangle$ afterward.

Let Alice measure $|\rightarrow\rangle$, $|\leftarrow\rangle$ basis. Bob's system will be in state $|\leftarrow\rangle$ or $|\rightarrow\rangle$ afterward.

**Conclusion**

$|\Psi\rangle$ is information.
Application 1:
Quantum Teleportation
(Bennett, Brassard, Crépeau, Jozsa, Peres, Wootters)
$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$
from Charlie

\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \]
from Charlie

\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \]
\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \]
A Tool

Entangled Measurements

\[ \mathcal{H} = \sum_i \lambda_i |e_i><e_i| \]

\( |e_i\rangle \) — entangled vectors
Example: The Bell Basis

The states

$|\Psi^-\rangle = \frac{1}{\sqrt{2}} (|0\rangle|1\rangle - |1\rangle|0\rangle)$

$|\Psi^+\rangle = \frac{1}{\sqrt{2}} (|0\rangle|1\rangle + |1\rangle|0\rangle)$

$|\Xi^-\rangle = \frac{1}{\sqrt{2}} (|0\rangle|0\rangle - |1\rangle|1\rangle)$

$|\Xi^+\rangle = \frac{1}{\sqrt{2}} (|0\rangle|0\rangle + |1\rangle|1\rangle)$

form an orthonormal basis for $\mathcal{H}_2 \otimes \mathcal{H}_2$.

They are all nicely entangled!
Teleportation

\[ |\Psi^+\rangle \]

Charlie

Alice

Bob
Teleportation

\[ |\psi\rangle |\psi^-\rangle = (\alpha |0\rangle + \beta |1\rangle) \frac{1}{\sqrt{2}} (|0\rangle |1\rangle - |1\rangle |0\rangle) \]

\[ = \frac{\alpha}{\sqrt{2}} (|10\rangle |01\rangle - |10\rangle |10\rangle) + \frac{\beta}{\sqrt{2}} (|11\rangle |01\rangle - |11\rangle |10\rangle) \]

\[ = \frac{1}{2} |\psi^-\rangle (-\alpha |0\rangle - \beta |1\rangle) \]
\[ + \frac{1}{2} |\psi^+\rangle (-\alpha |0\rangle + \beta |1\rangle) \]
\[ + \frac{1}{2} |\Xi^-\rangle (\beta |0\rangle + \alpha |1\rangle) \]
\[ + \frac{1}{2} |\Xi^+\rangle (-\beta |0\rangle + \alpha |1\rangle) \]

\[ = \frac{1}{2} |\psi^-\rangle (-I |\psi\rangle) + \frac{1}{2} |\psi^+\rangle (-\sigma_x |\psi\rangle) \]
\[ + \frac{1}{2} |\Xi^-\rangle (\sigma_x |\psi\rangle) + \frac{1}{2} |\Xi^+\rangle (i\sigma_y |\psi\rangle) \]
Note:

1) Alice gains no information about $\alpha, \beta$.

2) Teleportation not complete until Bob receives classical information. (Only 2 bits!)

3) Entanglement itself can be teleported.
Summary

Entanglement

Good!
Belief Teleportation?


Charlie has partial knowledge of C knows A & B correlated

\[ p(H_c) = 1 - p(T_c) = \chi \]
\[ p(H_A, H_B) = p(T_A, T_B) = \frac{1}{2} \]

Teleportation:

1) Alice checks parity, announces result to Bob

2) Bob turns coin over if parity odd; otherwise does nothing

Upshot: \[ p(H_c) \rightarrow p(H_B) \]
Weatherman:
How does learning today’s weather modify his expectations for tomorrow’s?

Quantum Observer:
How does learning the consequence of this measurement interaction modify his expectations for the consequence of that measurement interaction?
Objects of Interest

Weatherman:
Ignorance of what is.

Quantum Observer:
Ignorance of what would come about if.

“the” quantum state
"Measurement"

Does it reveal a pre-existing, but unknown, value?

or

Does it in some sense go toward creating the very value?
"If, without in any way disturbing a system [one can gather the information required to] predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."
Motivated by EPR

Consider two spatially separated qutrits in a maximally entangled state:

\[ |EPR\rangle = \sum_{i=1}^{3} |i_i\rangle |i_i\rangle \]

Assume locality.

Now measure the left one any way you like. Say with A or B, two nondegenerate noncommuting observables.
So measurement is simple revelation after all?

If here, can predict there.

If here, can predict there.

Element of reality

Element of reality
EPR Still Implodes

But must consider many more bases than two. (\sim 44-46)

\[
\begin{align*}
\text{element of reality} & \quad \Rightarrow \\
\text{element of reality} & \quad \Rightarrow \\
\text{element of reality} & \quad \Rightarrow
\end{align*}
\]

Until contradiction.

(Hint, think of Kochen-Specker.)
Kochen-Specker

Suppose they did pre-exist.

Then we should be able to color every set of orthogonal rays in $\mathbb{R}^3$ red-green-green.
Kochen-Specker

Cannot be colored:

33 rays, Peres

(when completed into full triads, consists of 40 triads made from 57 rays)
### World Competition

#### $d = 3$

<table>
<thead>
<tr>
<th>$n$</th>
<th>Who</th>
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<tbody>
<tr>
<td>117</td>
<td>Kochen and Specker</td>
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<tr>
<td>109</td>
<td>Jost</td>
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<tr>
<td>33</td>
<td>Schütte</td>
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<td>33</td>
<td>Peres</td>
</tr>
<tr>
<td>33</td>
<td>Penrose</td>
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<td>31</td>
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#### $d = 4$

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<td>28</td>
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<td>20</td>
<td>Kernaghan</td>
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<td>18</td>
<td>Cabello, et.al.</td>
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#### $d = 5$

<p>| | |</p>
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</table>
Contradiction!

9 is not even.

Values would then necessitate an even result.

But each ray appears twice, preexistence of

Summing the values gives 9.

So, in each column one ray will be assigned 1, and

Each column represents an orthonormal basis.

| 1100 1010 1001 1101 0100 1111 1110 | 1100 1010 1001 1101 0100 1111 1110 |
| 1100 1010 1001 1101 0100 1111 1110 | 1100 1010 1001 1101 0100 1111 1110 |
| 1100 1010 1001 1101 0100 1111 1110 | 1100 1010 1001 1101 0100 1111 1110 |
| 1100 1010 1001 1101 0100 1111 1110 | 1100 1010 1001 1101 0100 1111 1110 |
| 1100 1010 1001 1101 0100 1111 1110 | 1100 1010 1001 1101 0100 1111 1110 |
| 1100 1010 1001 1101 0100 1111 1110 | 1100 1010 1001 1101 0100 1111 1110 |
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| 1100 1010 1001 1101 0100 1111 1110 | 1100 1010 1001 1101 0100 1111 1110 |
| 1100 1010 1001 1101 0100 1111 1110 | 1100 1010 1001 1101 0100 1111 1110 |

Cabelllo's Ray Proof in $\mathbb{R}$
Quantum measurements are generative:

Their outcomes do not pre-exist before the measurement interaction; they arise from the very process.
P(h)
$P(h)$

states of pre-existent reality

consequences of "measurement" interactions
The Pauli’an Idea, 1

[Einstein and I] often discussed these questions, and I invariably profited very greatly even when I could not agree with Einstein’s views. “Physics is after all the description of reality,” he said to me, continuing, with a sarcastic glance in my direction, “or should I perhaps say physics is the description of what one merely imagines?” This question clearly shows Einstein’s concern that the objective character of physics might be lost through a theory of the type of quantum mechanics, in that as a consequence of its wider conception of objectivity of an explanation of nature the difference between physical reality and dream or hallucination become blurred.

The objectivity of physics is however fully ensured in quantum mechanics in the following sense. Although in principle, according to the theory, it is in general only the statistics of series of experiments that is determined by laws, the observer is unable, even in the unpredictable single case, to influence the result of his observation—as for example the response of a counter at a particular instant of time. Further, personal qualities of the observer do not come into the theory in any way—the observation can be made by objective registering apparatus, the results of which are objectively available for anyone’s inspection. Just as in the theory of relativity a group of mathematical transformations connects all possible coordinate systems, so in quantum mechanics a group of mathematical transformations connects the possible experimental arrangements.

Einstein however advocated a narrower form of the reality concept …

— Wolfgang Pauli

“Albert Einstein and the Development of Physics,” 1958
What is a state vector?
(A. Peres, Am. J. Phys. 52, 644 (1984))

Information!

- Compared collapse to Bayesian updating
- Old Einstein argument that real states of affairs cannot be toggled from a distance
- Demystified quantum teleportation

Unperformed experiments have no results
(A. Peres, Am. J. Phys. 46, 745 (1978))

Amen!

- Gleason’s Theorem with POVMs
- EPR criterion of reality fails
- Cabello’s Kochen-Specker construction
Shannon Information

The "information" we gather in an experiment depends upon our prior expectations.

\[\begin{align*}
p(\bullet) &= \frac{1}{2} \\
p(\circ) &= \frac{1}{2} \quad \text{a lot to be gained}
\end{align*}\]

\[\begin{align*}
p(\bullet) &= 0.90 \\
p(\circ) &= 0.10 \quad \text{little to be gained}
\end{align*}\]
**Shannon Information**

set of discrete probabilities

\[ \Gamma_m = \{ (p_1, \ldots, p_m) \mid p_j \geq 0, \sum_{j=1}^{m} p_j = 1 \} \]

\[ \Gamma = \bigcup_{m} \Gamma_m \]

information \( H : \Gamma \to \mathbb{R}_+ \)

1) \( H(A, B) \leq H(A) + H(B) \)

2) \( H(A, B) = H(A) + H(B) \) if \( A \leftrightarrow B \)

3) \( H(p_1, p_2, \ldots, p_m) \) — perm. invariant

4) \( H(p_1, \ldots, p_m, 0) = H(p_1, \ldots, p_m) \)

5) \( \lim_{p \to 0} H(p, 1-p) = 0 \)

\[ H(p_1, \ldots, p_m) = -C \sum_{j=1}^{m} p_j \log_2 p_j \]
Classical Information

\[ I(X,Y) = H(X) + H(Y) - H(X,Y) \]

\[ = H(Y) - H(Y|X) \]

\[ H(X,Y) = -\sum_{x,y} p(x,y) \log p(x,y) \]

\[ H(Y|X) = \sum_x p(x) H(Y|x) \]

\[ = -\sum_{x,y} p(x)p(y|x) \log p(y|x) \]
If A and B are entangled, measurements on A can reveal an "unnatural" amount of information about B.
Type-II Parametric Down Conversion

\[ |\psi\rangle = |\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle \]

Always one photon passes and one gets absorbed regardless of \( \theta \).
What’s Unnatural
About That?

Maybe A and B just big instruction sets?

If $\theta = 30^\circ$, P.
If $\theta = 31^\circ$, A.
If $\theta = 32^\circ$, A.

If $\theta = 30^\circ$, A.
If $\theta = 31^\circ$, P.
If $\theta = 32^\circ$, P.
Oh yeah?!

Consider variables

\[
\angle ab' = \angle b'a' = \angle a'b = \theta/3.
\]

EPR: \( a, b', a', b \in \mathbb{R}^+,-3 \), we just don't know which.

I.e. \( p(a, b', a', b) \) exists!

But, "information" in \( p(\cdots, \cdots, \cdots) \) not consistent with quantum mechanics.
Bell Theorems

Assume \( H(A, B', A', B) \) exists.

\[
H(A, B', A', B) = H(B) + H(A' | B) + H(B' | A', B) + H(A | B', A', B) \\
\leq H(B) + H(A' | B) + H(B' | A') + H(A | B')
\]

But

\[
H(A, B) \leq H(A, B', A', B).
\]

With

\[
H(A | B) = H(A, B) - H(B)
\]

\[\Rightarrow\]

\[
0 \leq H(A | B') + H(B' | A') + H(A' | B) - H(A | B)
\]

Not true for small \( \theta \)!
Unknown States

We try to clone them:

\[ |\psi\rangle \rightarrow |\psi\rangle |\psi\rangle \]

We teleport them:

\[ |\psi\rangle \rightarrow \text{2 bits} \rightarrow |\psi\rangle \]

We protect them:

\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \]

\[ \rightarrow \alpha (|000\rangle + |111\rangle)^{\otimes 3} \]
\[ + \beta (|000\rangle - |111\rangle)^{\otimes 3} \]
Unknown States?
Quantum State Tomography
Tomography on a Qubit

Operator space is a linear vector space in its own right.

\((\hat{A}, \hat{B}) = \text{tr} \hat{A}^\dagger \hat{B} \) — inner product

If state is \(\hat{\Pi} = |\psi\rangle \langle \psi|\), “projections”

\[1 = |\langle \psi | \psi \rangle|^2 = \text{tr} \hat{\Pi} \hat{1}\]

\[\overline{\sigma}_x = \langle \psi | \hat{\sigma}_x | \psi \rangle = \text{tr} \hat{\Pi} \hat{\sigma}_x\]

\[\overline{\sigma}_y = \langle \psi | \hat{\sigma}_y | \psi \rangle = \text{tr} \hat{\Pi} \hat{\sigma}_y\]

\[\overline{\sigma}_z = \langle \psi | \hat{\sigma}_z | \psi \rangle = \text{tr} \hat{\Pi} \hat{\sigma}_z\]

fix the state uniquely.

\(\hat{1}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\) — linearly indep.
Quantum State Tomography
Quantum State Tomography

Essence is that $\hat{\rho}$ evolve toward $\hat{\rho} \otimes \hat{\rho} \otimes \hat{\rho} \ldots$ with mmt.

nothing left to be "learned"
Condition for Tomography

\[ \rho^{(n)} \rightarrow \hat{\rho}^{(n)} \]
A Quantum de Finetti?

Can exchangeability (suitably defined) exorcise box boy?

Candidates:

\( \hat{\rho}^{(n)} \in \mathcal{B}(\mathcal{H}^\otimes n) \) is n-exchangeable if permutation invariant

\( \{ \hat{\rho}^{(n)} \}_{n=1}^\infty \) is an exchangeable sequence if

1) \( \hat{\rho}^{(n)} = \text{tr}_{n+1} \hat{\rho}^{(n+1)} \ \forall n \)

2) \( \hat{\rho}^{(n)} \) n-exchangeable \( \forall n \)

Theorem? \( \hat{\rho}^{(n)} \) exchangeable sequence iff

\( \hat{\rho}^{(n)} = \int P(\hat{\rho}) \hat{\rho}^\otimes n \ d\hat{\rho} \) ?
Bureau of Standards

the "standard" quantum measurement
Bureau of Standards

the "standard" quantum measurement

p(h)
Standard measurements not good enough for the bureau.

\[ H = \sum_i \alpha_i \Pi_i \quad , \quad \Pi_i = |i\rangle\langle i| \]

\[ p(i) = \text{tr} \rho \Pi_i = \langle i| \rho |i\rangle \]

\[ \Rightarrow \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots \\ \rho_{21} & \rho_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \]
Informational Completeness

quantum states

$\rho \in \mathcal{L}(\mathcal{H}_d) \quad -$ D$^2$-dimensional vector space

Choose POVM $\{E_n\}$, $h=1,\ldots, D^2$, with $E_n$ all linearly independent. (Can be done.)

$D^2$ numbers $\rho(h) = \text{tr}\rho E_n$ determine $\rho$.

Any $\{E_n\}$ can be the standard quantum measurement.
Path Back to Density Ops

Suppose \( \{E_j\}, j=1, \ldots, d^2, \) is ICp.

Then \( \rho(j) \) determines \( \rho \).
But also \( \rho = \sum_j \alpha_j E_j \) for some \( \alpha_j \)'s.

Thus

\[
\rho(j) = \text{tr} \rho E_j = \sum_k \alpha_k \text{tr} E_j E_k
\]
i.e.

\[
\vec{\alpha} = M \vec{\rho}
\]
where \( M = [\text{tr} E_j E_k] \)
and so

\[
\vec{\alpha} = M^{-1} \vec{\rho}
\]

Prettiest when \( M_{jk} = a + b \delta_{jk} \).
A Very Fundamental Moment?

Suppose $d^2$ projectors $\Pi_i = |\psi_i\rangle\langle \psi_i|$ satisfying

$$\text{tr} \, \Pi_i \Pi_j = \frac{1}{d+1}, \quad i \neq j$$

exist.

Can prove:

1) the $\Pi_i$ linearly independent

2) $\sum_i \frac{1}{d} \Pi_i = I$

So good for Bureau of Standards.

Also

$$\rho(i) = \frac{1}{d} \text{tr} \rho \Pi_i$$

$$\rho = \sum_i \left[ (d+1)\rho(i) - \frac{1}{d} \right] \Pi_i$$
Evidence for Existence

Analytical Constructions

$$d = 2 - 13, 15, 19$$

Numerical ($$e \leq 10^{-9}$$)

$$d = 2 - 47$$
Pure States in SIC Language

Conditions

$$\rho^+ = \rho \quad , \quad \text{tr} \rho^2 = \text{tr} \rho^3 = 1$$

translate to

$$\sum_i \rho(i)^2 = \frac{2}{d(d+1)}$$

and

$$\sum_{jkl} c_{jkl} \rho(j) \rho(k) \rho(l) = \frac{d+7}{(d+1)^3}$$

where

$$c_{jkl} = \text{Re} \ \text{tr} \ \Pi_j \Pi_k \Pi_l$$

Could these be independently motivatable physical constants?
Measure observable \{ P_j \}.

Probability of outcome \( j \) given by

\[
\rho(j) = \text{tr} \rho P_j
\]

"The Born Rule"
Laws of Probability

$H_i$ - various hypotheses one might have

$D_j$ - data values one might gather

**Given:**

$p(D_j | H_i) \leftrightarrow \text{expectations for data given hypothesis}$

$p(H_i) \leftrightarrow \text{expectations for hypotheses themselves}$

**Question:** What expectations should one have for the $D_j$?

**Answer:** $P(D_j) = \sum_i p(H_i) p(D_j | H_i)$
What $p(D_j)$?

But really going to do this.

Given $p(H_i)$

Any SIC

Given $p(D_j|H_i)$

Any von Neumann measurement
\[ p(D_j) = (d+1) \sum_i p(H_i) p(D_j | H_i) - 1 \]

Quantum

(Usual) Bayesian

Magic!
Unitarity

\[ \rho \rightarrow u \rho u^+ \]
\[ p(i) \rightarrow q(j) \]

Define \( t(j|i) = \frac{1}{d} \text{tr} \ U \Pi_i U^+ \Pi_j \)

doubly stochastic matrix

Then

\[ q(j) = (d+1) \sum_i p(i) t(j|i) - \frac{1}{d} \]

Could hardly be simpler.
## Tegmark Poll

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<th>Interpretation</th>
<th>Votes</th>
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<td>Many Worlds</td>
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<td>Bohm</td>
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<td>Consistent Histories</td>
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<td>Modified Dynamics (GRW)</td>
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<tr>
<td>None of the above/undecided</td>
<td>18</td>
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**Axioms: Quantum**

0) Systems exist.

1) Associated with each is a complex vector space $\mathcal{H}$.

2) Measurements correspond to orthonormal bases $|e_i\rangle$ on $\mathcal{H}$.

3) States correspond to density operators $\rho$ on $\mathcal{H}$.

4) Systems combine by tensor product of their vector spaces, $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

5) When no measurement is performed, states evolve by unitary maps $U$. 
Special Relativity

c is constant.

Physics is constant.
What is real about a system?
I want you to frame a question, as sharp and clear as possible—one to which you do not yet know the answer, but desperately want to know, and expect someday to know.

Pretend to be David Hilbert. The Millennium is approaching. Issue a challenge to the quantum theorists of the 21st century. List the key questions they should seek to answer. Hard questions, but not hopelessly hard, questions whose answers could transform our understanding of how the physical world works.

— John Preskill

12 August 1998
Two graduate scholarships available for work with me at University of Waterloo; 4 year tenure.

If interested, write
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